#1

1970 AB4
Solution

Method 1:

The combined surface area of the hemisphere and its base is
$S = \frac{1}{2}(4\pi r^2) + \pi r^2 = 3\pi r^2$

$$18 = \frac{dS}{dt} = 6\pi r \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{3}{\pi r}$$

Since the height of the cone is $h = r$, the volume of the cone is $V = \frac{1}{3}\pi r^3$

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt} = \pi r^2 \left(\frac{3}{\pi r}\right) = 3r$$

At $r = 4$, $\frac{dV}{dt} = 12$

Method 2:

As in method 1, $S = 3\pi r^2$ and so $V = \frac{1}{3}\pi\left(\frac{S}{3\pi}\right)^{\frac{3}{2}}$.

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot \frac{3}{2}\left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \cdot \frac{1}{3\pi} \cdot \frac{dS}{dt} = \frac{1}{6}\left(\frac{S}{3\pi}\right)^{\frac{1}{2}} \frac{dS}{dt}$$

When $r = 4$, $S = 48\pi$ and so $\frac{dV}{dt} = \frac{1}{6} \cdot 4 \cdot 18 = 12$

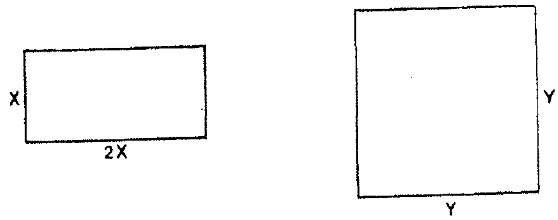
1972 - AB 4, BC 3

#2

4. A man has 340 yards of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.

(a) If x is the width of the rectangular field, what are the maximum and minimum possible values of x ?

(b) What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.



a.) $6x + 4y = 340$

$2x^2 \geq 800 \Rightarrow x \geq 20$

$y^2 \geq 100 \Rightarrow y \geq 10$

Since $y = \dfrac{340 - 6x}{4}$, $85 - \frac{3}{2}x \geq 10 \Rightarrow 50 \geq x$

Therefore, min of x is 20 and max of x is 50.

b.) $A = 2x^2 + y^2$

$A = 2x^2 + (85 - \frac{3}{2}x)^2, \quad 20 \leq x \leq 50$

$$\frac{dA}{dx} = 4x + 2(85 - \tfrac{3}{2}x)(-\tfrac{3}{2})$$
$$= 4x + \tfrac{9}{2}x - 255$$
$$= \tfrac{1}{2}(17x - 510)$$

$\therefore \frac{dA}{dx} = 0$ when $17x = 510$

or $x = 30$

CLOSED INTERVAL TEST:

x	$A(x)$
20	$800 + (55)^2 = 3825$
30	$1800 + (45)^2 = 3400$
50	$5000 + (10)^2 = 5100$ (ABSOLUTE MAX)

$\therefore$ The greatest number of square yards that can be enclosed in the two fields is 5100.

1977 – AB 6

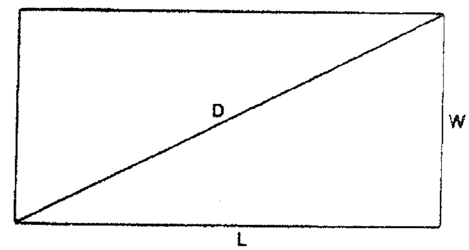
#3

6. A rectangle has a constant area of 200 square meters and its length L is increasing at the rate of 4 meters per second.

(a) Find the width W at the instant the width is decreasing at the rate of 0.5 meters per second.

(b) At what rate is the diagonal D of the rectangle changing at the instant when the width W is 10 meters?

Given: $LW = 200, \ \frac{dL}{dt} = 4$



a.) Find W when $\frac{dW}{dt} = -.5$

$$L\frac{dW}{dt} + W\frac{dL}{dt} = 0$$

$$\frac{200}{W}(-.5) + W(4) = 0$$

$$\frac{-100 + 4W^2}{W} = 0 \Rightarrow W = 5 \text{ and } L = 40$$

b) Find $\frac{dD}{dt}$ when $W = 10$

$$D^2 = L^2 + W^2$$

$$\cancel{2}D\frac{dD}{dt} = \cancel{2}L\frac{dL}{dt} + \cancel{2}W\frac{dW}{dt}$$

When $W = 10$, $L = 20$, $D = 10\sqrt{5}$,

From part a:

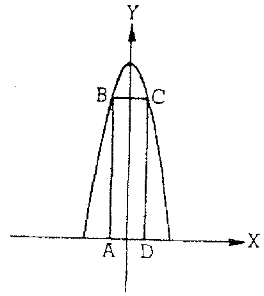
$$\frac{dW}{dt} = \frac{-W\frac{dL}{dt}}{L}$$

$$\frac{dW}{dt} = -\frac{W}{L}(4)$$

and $10\sqrt{5}\frac{dD}{dt} = 20(4) + 10\left(\frac{-10}{20}\cdot 4\right)$

$$\frac{dD}{dt} = \frac{80 - 20}{10\sqrt{5}} = \frac{6}{\sqrt{5}}$$

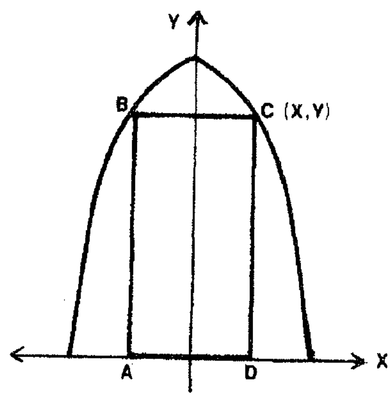
\#4 1980 - AB2



2. A rectangle ABCD with sides parallel to the coordinate axes is inscribed in the region enclosed by the graph of $y = -4x^2 + 4$ and the X-axis as shown in the figure above.

(a) Find the x- and y-coordinates of C so that the area of rectangle ABCD is a maximum.

(b) The point C moves along the curve with its x-coordinate increasing at the constant rate of 2 units per second. Find the rate of change of the area of rectangle ABCD when $x = \frac{1}{2}$.



a.) $A = 2xy = 2x(-4x^2+4), \quad 0 \le x \le 1$

$A = -8x^3 + 8x$

$\frac{dA}{dx} = -24x^2 + 8$

$\frac{dA}{dx} = 0$ when $x^2 = \frac{1}{3}$ or $x = \frac{1}{\sqrt{3}}$

$\frac{dA}{dx}$: 0 $+$ $\frac{1}{\sqrt{3}}$ $-$ 1 x $\implies$ A has absol. max. at $x = \frac{1}{\sqrt{3}}$

$\boxed{\text{Coordinates of } C\text{: } \left(\frac{\sqrt{3}}{3}, \frac{8}{3}\right)}$

b) $\frac{dx}{dt} = 2$. Find $\frac{dA}{dt}$ when $x = \frac{1}{2}$

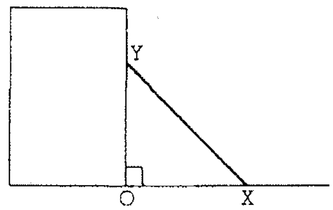
$A = -8x^3 + 8x \implies \frac{dA}{dt} = -24x^2 \frac{dx}{dt} + 8\frac{dx}{dt}$

$\frac{dA}{dt} = (-24)(\frac{1}{4})(2) + 8(2) = \boxed{4}$

∴ A is changing at the rate of 4 unit²/sec.

1982 - AB4

#5



4. A ladder 15 feet long is leaning against a building so that end X is on level ground and end Y is on the wall as shown in the figure. X is moved away from the building at the constant rate of $\frac{1}{2}$ foot per second.

(a) Find the rate in feet per second at which the length OY is changing when X is 9 feet from the building.

(b) Find the rate of change in square feet per second of the area of triangle XOY when X is 9 feet from the building.

$$x^2 + y^2 = 15^2$$

Implicit

a.) $\frac{dx}{dt} = \frac{1}{2}$

$\cancel{2}x \frac{dx}{dt} + \cancel{2}y \frac{dy}{dt} = 0$

$9\left(\frac{1}{2}\right) + 12 \frac{dy}{dt} = 0$

$\boxed{\frac{dy}{dt} = -\frac{3}{8}}$

Explicit

$y = \sqrt{225 - x^2}$

$\frac{dy}{dt} = \frac{-x \frac{dx}{dt}}{\sqrt{225-x^2}}$

$\frac{dy}{dt} = \frac{-9\left(\frac{1}{2}\right)}{12} = -\frac{3}{8}$

b.) **Implicit**

$A = \frac{1}{2} xy$

$\frac{dA}{dt} = \frac{1}{2}\left(x \frac{dy}{dt} + y \frac{dx}{dt}\right)$

$\left.\frac{dA}{dt}\right|_{x=9} = \frac{1}{2}\left(9\left(-\frac{3}{8}\right) + 12\left(\frac{1}{2}\right)\right)$

$\boxed{\frac{dA}{dt} = \frac{21}{16}}$

Explicit

$A = \frac{1}{2} x \sqrt{225 - x^2}$

$\frac{dA}{dt} = \frac{1}{2}\left(\sqrt{225-x^2} \frac{dx}{dt} + x \cdot \frac{-2x}{2\sqrt{225-x^2}} \cdot \frac{dx}{dt}\right)$

$\left.\frac{dA}{dt}\right|_{x=9} = \frac{1}{2}\left(6 - \frac{27}{8}\right) = \frac{21}{16}$

1984 - AB 5

#6

5. The volume V of a cone $\left(V = \frac{1}{3}\pi r^2 h\right)$ is increasing at the rate of 28π cubic units per second. At the instant when the radius r of the cone is 3 units, its volume is 12π cubic units and the radius is increasing at $\frac{1}{2}$ unit per second.

(a) At the instant when the radius of the cone is 3 units, what is the rate of change of the area of its base?

(b) At the instant when the radius of the cone is 3 units, what is the rate of change of its height h?

(c) At the instant when the radius of the cone is 3 units, what is the instantaneous rate of change of the area of its base with respect to its height h?

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a.) $A_{base} = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 3 \cdot \frac{1}{2}$$

$$\boxed{\frac{dA}{dt} = 3\pi}$$

b.) $$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + h \cdot \frac{2}{3}\pi r \frac{dr}{dt}$$

$$28\pi = \frac{1}{3}\pi \cdot 9 \cdot \frac{dh}{dt} + 4 \cdot \frac{2}{3}\pi \cdot 3 \cdot \frac{1}{2}$$

$$\frac{24}{3} = \frac{dh}{dt}$$

$$\boxed{\frac{dh}{dt} = 8}$$

$V = \frac{1}{3}\pi r^2 h$

$12\pi = \frac{1}{3}\pi \cdot 3^2 \cdot h$

$4 = h$

c.) $$\frac{dA}{dh} = \frac{\frac{dA}{dt}}{\frac{dh}{dt}}$$

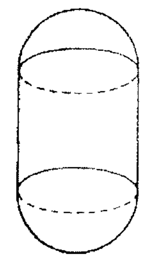
$$\boxed{\frac{dA}{dh} = \frac{3\pi}{8}}$$

#7 1985 - AB5, BC2

5. The balloon shown above is in the shape of a cylinder with hemispherical ends of the same radius as that of the cylinder. The balloon is being inflated at the rate of 261π cubic centimeters per minute. At the instant the radius of the cylinder is 3 centimeters, the volume of the balloon is 144π cubic centimeters and the radius of the cylinder is increasing at the rate of 2 centimeters per minute. (The volume of a cylinder with radius r and height h is $\pi r^2 h$, and the volume of a sphere with radius r is $\frac{4}{3}\pi r^3$.)

(a) At this instant, what is the height of the cylinder?

(b) At this instant, how fast is the height of the cylinder increasing?



Given: $\frac{dV}{dt} = 261\pi$;

$V = 144\pi$ and $\frac{dr}{dt} = 2$ when $r = 3$

a.) Find h when $r = 3$

$$V = \pi r^2 h + \frac{4}{3}\pi r^3; \quad 144\pi = 9\pi h + \frac{4}{3}\pi(27) \Rightarrow$$

$$9\pi h = 108\pi \Rightarrow h = 12 \text{ when } r = 3$$

b.) $$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + \pi h(2r)\frac{dr}{dt} + \frac{4}{3}(3\pi r^2)\frac{dr}{dt}$$

$$261\pi = 9\pi\frac{dh}{dt} + \pi(12)(6)(2) + 4(9\pi)2$$

$$9\pi\frac{dh}{dt} = 45\pi \Rightarrow \frac{dh}{dt} = 5$$

$\therefore$ h is increasing 5 cm/min when $r = 3$

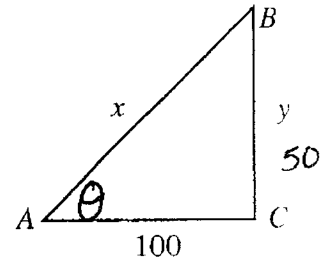
#8

AP 1988 – BC 3

The figure below represents an observer at point A watching balloon B as it rises from point C. The balloon is rising at a constant rate of 3 meters per second and the observer is 100 meters from the point C.

$$\frac{dy}{dt} = 3 \text{ m/s}$$

$$100^2 + 50^2 = x^2$$
$$x = \sqrt{12500}$$



a. Find the rate of change in x at the instant when $y = 50$.

b. Find the rate of change ion the area of right triangle BCA at the instant when $y = 50$.

c. Find the rate if change θ at the instant when $y = 50$.

a) $x^2 = 100^2 + y^2$

$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$

$2\sqrt{12500}\,\frac{dx}{dt} = 2(50)(3)$

$\boxed{\frac{dx}{dt} = \frac{150}{\sqrt{12500}} \text{ m/s}}$

b) $A = \frac{1}{2}(100)y \quad A = 50y$

$\frac{dA}{dt} = 50\frac{dy}{dt}$

$\frac{dA}{dt} = 50(3)$

$\boxed{\frac{dA}{dt} = 150 \text{ m}^2/\text{s}}$

c) $\tan\theta = \frac{y}{100}$

$100\tan\theta = y$

$100\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{dy}{dt}$

$100\left(\frac{1}{\cos\theta}\right)^2 \cdot \frac{d\theta}{dt} = \frac{dy}{dt}$

$100\left(\frac{1}{100/\sqrt{12500}}\right)^2 \cdot \frac{d\theta}{dt} = 3$

$100\left(\frac{12500}{10000}\right) \cdot \frac{d\theta}{dt} = 3$

$125 \cdot \frac{d\theta}{dt} = 3$

$\boxed{\frac{d\theta}{dt} = \frac{3}{125} \text{ rad./s}}$