Calculus with Parametrics

parametric equations $x = f(t)$, $y = g(t)$

$$\text{slope} = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

EX1 $x = \sin t$
$y = \cos t$

A. Find $\frac{dy}{dx}$ when $t = \pi/4$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{\cos t} = -\tan t \Big|_{t=\pi/4} = -\tan \pi/4 = -1$$

B. Write the eqn. of the tangent line to the curve when $t = \pi/4$.

point $(x, y) = (\sin \pi/4, \cos \pi/4) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

slope $= -1$

$$y - \frac{\sqrt{2}}{2} = -1\left(x - \frac{\sqrt{2}}{2}\right)$$

2nd derivative

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}}$$

EX2 $x = \sqrt{t} = t^{1/2}$

$y = \frac{1}{4}(t^2 - 4)$ for $t \geq 0$.

Find the slope & concavity at $\overset{x\ \ y}{(2,3)}$.

Slope $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{4}\cdot 2t^1}{\frac{1}{2}\cdot t^{-1/2}} = t^{3/2}\Big|_{\substack{(2,3)\\ t=4}} = 4^{3/2} = 8$$

$x = \sqrt{t}$

$2 = \sqrt{t}$

$t = 4$

concavity

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{d(y')}{dt}}{\frac{dx}{dt}} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}} = 3t\Big|_{t=4} = 12$$

concave up

EX3 $x = \cos\theta$

$y = 3\sin\theta$

Find the slope & concavity when $\theta = 0$.

Slope $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos\theta}{-\sin\theta} = -3\cot\theta\Big|_{\theta=0} = \text{undefined}$$

tangent line is vertical when $\theta = 0$

concavity

$$\frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{\frac{d(y')}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3(-\csc^2\theta)}{-\sin\theta} = -3\csc^3\theta$$

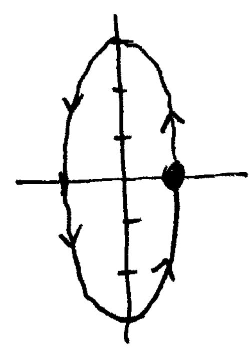
or

$$\left.\frac{-3}{\sin^3\theta}\right|_{\theta=0}$$

undefined

~~possible~~ point of inflection when $\theta = 0$

point $(1, 0)$



Arc Length with Parametrics

$$a.l. = S = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex 4 $x = 5\cos t - \cos(5t)$

$y = 5\sin t - \sin(5t)$

Find the arc length over $[0, \frac{\pi}{2}]$.

$$a.l. = \int_0^{\pi/2} \sqrt{(-5\sin t + 5\sin(5t))^2 + (5\cos t - 5\cos(5t))^2}\, dt$$

$$= \boxed{10}$$