Comparison Tests — series must have pos. terms

* idea is to pick a similar, simpler series to compare with the given series

usually p-series or geometric

Direct Comparison Test

Let $0 < a_n < b_n$.

① If $\sum b_n$ ← bigger converges, then $\sum a_n$ ← smaller converges.

② If $\sum a_n$ ← smaller diverges, then $\sum b_n$ ← bigger diverges.

pick

given

EX 1 Determine convergence.

A. $\sum \dfrac{1}{n^4+1}$ pick $\sum \dfrac{1}{n^4}$ p-series $p = 4$ $4 > 1$ converges by the p-series test

$\dfrac{1}{n^4+1} < \dfrac{1}{n^4}$ ← bigger converges

then $\sum \dfrac{1}{n^4+1}$ converges by the direct comp. test

B. $\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{5}} + \dfrac{1}{\sqrt{8}} + \dots + \dfrac{1}{\sqrt{3n-1}} + \dots$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n-1}}$ pick $\displaystyle\sum \frac{1}{\sqrt{3n}} = \sum \frac{1}{\sqrt{3}\cdot\sqrt{n}} = \frac{1}{\sqrt{3}} \sum \frac{1}{n^{1/2}}$

p-series $p = \frac{1}{2}$ $\frac{1}{2} < 1$

series div. by p-series test

$$\frac{1}{\sqrt{3n}} < \frac{1}{\sqrt{3n-1}}$$

smaller div.

$\sum \frac{1}{\sqrt{3n-1}}$ div. by the direct comparison test

(c) $\sum \frac{3^n}{4^n+5}$ pick $\sum \frac{3^n}{4^n} = \sum \left(\frac{3}{4}\right)^n$ geom. $r = \frac{3}{4} < 1$

conv. by the geometric series test

$$\frac{3^n}{4^n+5} < \frac{3^n}{4^n}$$

bigger conv.

$\sum \frac{3^n}{4^n+5}$ conv. by the Direct Comparison Test

(D) $\sum \frac{1}{3+\sqrt{n}}$ pick $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ p-series $p = \frac{1}{2}$ $\frac{1}{2} < 1$ div. by p-series test

$$\frac{1}{3+\sqrt{n}} < \frac{1}{\sqrt{n}}$$

← bigger div.

pick $\sum \frac{1}{n}$ p-series $p = 1$ div. by the p-series test

$$\frac{1}{n} < \frac{1}{3+\sqrt{n}}$$

smaller div.

$\sum \frac{1}{3+\sqrt{n}}$ div. by the Direct Comp. Test

Limit Comparison Test

If $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ is a positive & finite #, (a_n ← given, b_n ← pick)

then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Ex 2

A. $\sum \frac{1}{3+\sqrt{n}}$ pick $\sum \frac{1}{\sqrt{n}}$ p-series $p = 1/2$ $\frac{1}{2} < 1$ div.

$$\lim_{n\to\infty} \frac{\frac{1}{3+\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{3+\sqrt{n}} = \frac{1}{1} = 1 \quad \text{← pos. \& finite}$$

$\sum \frac{1}{3+\sqrt{n}}$ diverges by the Limit Comp. Test

B. $\sum \frac{1}{2n+1}$ pick $\sum \frac{1}{2n} = \sum \frac{1}{2n} = \frac{1}{2}\sum \frac{1}{n}$

p-series $p = 1$ div. by p-series test

$$\lim_{n\to\infty} \frac{\frac{1}{2n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} \text{ pos. \& finite}$$

$\sum \frac{1}{2n+1}$ diverges by the limit comparison test

C. $\sum \frac{n^2-10}{4n^5+3n^2}$ pick $\sum \frac{n^2}{n^5} = \sum \frac{1}{n^3}$

p-series $p=3$ $3>1$
conv. by p-series test

$$\lim_{n\to\infty} \left(\frac{\frac{n^2-10}{4n^5+3n^2}}{\frac{1}{n^3}} \right) = \lim_{n\to\infty} \frac{n^5-10n^3}{4n^5+3n^2} = \frac{1}{4} \text{ pos \& finite}$$

$\sum \frac{n^2-10}{4n^5+3n^2}$ converges by Limit Comparison Test

D. $\sum \tan\left(\frac{1}{n}\right)$ pick $\sum \frac{1}{n}$ p-series $p=1$ div.
p-series test

$$\lim_{n\to\infty} \frac{\tan\left(\frac{1}{n}\right)}{\frac{1}{n}} \quad \frac{0}{0} \quad \text{L'Hopital's}$$

$$\lim_{n\to\infty} \frac{\sec^2\left(\frac{1}{n}\right) \cdot \cancel{-\frac{1}{n^2}}}{\cancel{-\frac{1}{n^2}}} = \sec^2(0) = 1 \text{ pos \& finite}$$

$\sum \tan\left(\frac{1}{n}\right)$ diverges by the limit comp. test

order

1. n^{th} term
2. special series? (p-, geom., telesc.)
3. comparison (direct/ limit)
4. integral