Exponential Functions

$y=e^x$

domain: $(-\infty, \infty)$

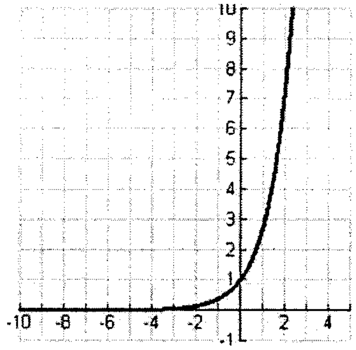
range: $(0, \infty)$

continuous on its domain

always increasing (strictly monotonic)

concave up

one-to-one



Derivative Rules

$$\frac{d}{dx}\left(e^x\right) = e^x$$

$$\frac{d}{dx}\left(a^x\right) = a^x \ln a$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$$\frac{d}{dx}\left(\log_a x\right) = \frac{1}{x\ln a}$$

Integration Rules

$$\int \frac{1}{x}dx = \ln|x| + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Example 1 **Find the derivative:**

A. $f(x) = e^{4x^2-6}$

$f'(x) = e^{4x^2-6} \cdot (8x) = 8xe^{4x^2-6}$

B. $f(x) = e^{-\frac{3}{x}}$ $-3x^{-1}$

$f'(x) = e^{-\frac{3}{x}} \cdot (3x^{-2}) = \dfrac{3}{x^2 e^{\frac{3}{x}}}$

C. $y = x^7 e^x$

$y' = x^7 \cdot e^x + e^x \cdot 7x^6$

$= x^6 e^x (x+7)$

D. $ye^x + 2x - 3y^2 = 10$

$$y \cdot e^x + e^x \cdot 1 \frac{dy}{dx} + 2 - 6y\frac{dy}{dx} = 0$$

$$e^x \frac{dy}{dx} - 6y\frac{dy}{dx} = -2 - ye^x$$

$$\frac{dy}{dx}(e^x - 6y) = -2 - ye^x \qquad \frac{dy}{dx} = \frac{-2 - ye^x}{e^x - 6y}$$

E. $y = 3^{4x}$

$$y' = 3^{4x} \cdot \ln 3 \cdot 4$$

Example 2 **Find:**

A. $\int e^{3x+1}\,dx$

$u = 3x+1$

$\frac{du}{dx} = 3$

$\frac{1}{3}du = dx$

$$\frac{1}{3}\int e^u\,du$$

$$\frac{1}{3}e^u + C$$

$$\boxed{\frac{1}{3}e^{3x+1} + C}$$

B. $\int \frac{e^{\frac{1}{x}}}{x^2}\,dx$

$u = \frac{1}{x} = x^{-1}$

$\frac{du}{dx} = -1x^{-2}$

$-du = x^{-2}dx$

$$-\int e^u\,du$$

$$-e^u + c$$

$$\boxed{-e^{\frac{1}{x}} + c}$$

C. $\int 5xe^{-x^2}\,dx$

$u = -x^2$

$\frac{du}{dx} = -2x$

$-\frac{1}{2}du = x\,dx$

$$5 \cdot -\frac{1}{2}\int e^u\,du$$

$$-\frac{5}{2}e^u + C$$

$$\boxed{-\frac{5}{2}e^{-x^2} + c}$$

D. $\int_0^1 \frac{e^x}{1+e^x}\,dx$

$u = 1+e^x$

$\frac{du}{dx} = e^x$

$du = e^x dx$

$$\int \frac{du}{u}$$

$$\ln|u| + c$$

$$\ln|1+e^x| + c \Big|_0^1$$

$$\ln(1+e^1) - \ln(1+e^0)$$

$$\ln(1+e) - \ln 2$$

$$\boxed{\ln\frac{1+e}{2}}$$

E. $\int \frac{2e^x - 2e^{-x}}{(e^x+e^{-x})^2}\,dx$ $= \int \frac{2(e^x - e^{-x})}{(e^x+e^{-x})^2}\,dx$

$u = e^x + e^{-x}$

$\frac{du}{dx} = e^x - e^{-x}$

$du = (e^x - e^{-x})dx$

$2\int \frac{du}{u^2} = 2\int u^{-2}\,du = -2 \cdot u^{-1} + C$

$\boxed{\frac{-2}{e^x+e^{-x}} + C}$

F. $\int 3^x\,dx$

$\boxed{\frac{3^x}{\ln 3} + C}$

Example 3 Find an equation of the tangent line to the graph of $f(x) = e^{3x}\ln x$ at the point $(1, 0)$.

$f'(x) = e^{3x} \cdot \frac{1}{x} + \ln x \cdot e^{3x} \cdot 3$

$f'(1) = e^3 \cdot \frac{1}{1} + \ln 1 \cdot e^3 \cdot 3 = e^3$

$y - 0 = e^3(x-1)$

Example 4 Find the particular solution that satisfies the conditions $f''(x) = \sin x + e^{2x}$, $f(0) = \frac{1}{4}$, and $f'(0) = \frac{1}{2}$.

$f'(x) = \int (\sin x + e^{2x})dx = -\cos x + \frac{1}{2}e^{2x} + C$

$\int e^{2x}\,dx$

$u = 2x$

$\frac{du}{dx} = 2$

$\frac{1}{2}du = dx$

$\frac{1}{2}\int e^u\,du$

$\frac{1}{2}e^u + C$

$\frac{1}{2} = -\cos(0) + \frac{1}{2}e^{2(0)} + C$

$\frac{1}{2} = -1 + \frac{1}{2} + C$

$1 = C$

$f'(x) = -\cos x + \frac{1}{2}e^{2x} + 1$

$f(x) = \int \quad dx$

$f(x) = -\sin x + \frac{1}{4}e^{2x} + x + C$

$\frac{1}{4} = -\sin(0) + \frac{1}{4}e^{2(0)} + 0 + C$

$\frac{1}{4} = \frac{1}{4} + C \qquad C = 0$

$\boxed{f(x) = -\sin x + \frac{1}{4}e^{2x} + x}$