Derivatives of Inverse Trig Functions

* rules are based on implicit differentiation

$$y = \sin^{-1} x$$
$$\sin y = \sin(\sin^{-1} x)$$
$$\sin y = x$$

$$\sin^2 y + \cos^2 y = 1$$
$$\cos^2 y = 1 - \sin^2 y$$
$$\cos y = \sqrt{1 - \sin^2 y}$$

deriv: $\cos y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$
$$= \frac{1}{\sqrt{1-x^2}}$$

$$y = \tan^{-1} x$$
$$\tan y = \tan(\tan^{-1} x)$$
$$\tan y = x$$

$$\frac{\sin^2 y + \cos^2 y}{\cos^2 y} = \frac{1}{\cos^2 y}$$
$$\tan^2 y + 1 = \sec^2 y$$

deriv: $\sec^2 y \dfrac{dy}{dx} = 1$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1}$$
$$= \frac{1}{x^2+1}$$

EX1 Find the derivative.

A. $y = \sin^{-1}(2x)$

$$y' = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

B. $f(x) = -2\cos^{-1}(3x-1) + 5x$

$$f'(x) = -2 \cdot -\frac{1}{\sqrt{1-(3x-1)^2}} \cdot 3 + 5 = \frac{6}{\sqrt{1-(3x-1)^2}} + 5$$

C. $y = \sec^{-1}(e^{2x})$

$$y' = \frac{1}{\cancel{|e^{2x}|}\sqrt{(e^{2x})^2-1}} \cdot \cancel{e^{2x}} \cdot 2 = \frac{2}{\sqrt{e^{4x}-1}}$$

EX2 $2x^3 - 3y^2 = 8$. Find $\frac{d^2y}{dx^2}$.

1st deriv: $6x^2 - 6y\frac{dy}{dx} = 0$

$$\frac{-6y\frac{dy}{dx}}{-6y} = \frac{-6x^2}{-6y}$$

$$\frac{dy}{dx} = \frac{x^2}{y}$$

2nd deriv. $\frac{d^2y}{dx^2} = \frac{(y)(2x) - x^2\left(1\frac{dy}{dx}\right)}{(y)^2} = \frac{2xy - x^2 \cdot \frac{x^2}{y}}{y^2}$

$$= \frac{\left(2xy - \frac{x^4}{y}\right) y}{\left(y^2\right) y} = \frac{2xy^2 - x^4}{y^3}$$

EX3 $x^2 - xy + y^2 = 7$

Find the eqns of the tangent and normal lines to the ellipse at $(-1, 2)$.

$$2x - \left[x \frac{dy}{dx} + y \cdot 1\right] + 2y \frac{dy}{dx} = 0$$

$$2x - x \frac{dy}{dx} - y + 2y \frac{dy}{dx} = 0$$

$$-x \frac{dy}{dx} + 2y \frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{-x + 2y}$$

$$\left.\frac{dy}{dx}\right|_{(-1,2)} = \frac{2 - 2(-1)}{-(-1) + 2(2)} = \frac{4}{5}$$

T: $y - 2 = \frac{4}{5}(x + 1)$

N: $y - 2 = -\frac{5}{4}(x + 1)$