Notes—Differential Equations Day 1 (Growth and Decay)

Steps to solving Separable differential equations

1. Look at the equation and identify the x's and y's
   - Treat dx as an x and dy as a y
2. Separate the x's and y's, so that they are on opposite sides of the equation.
   - It is very important that *one side of the equation is multiplied by dx and the other by dy*. Note: dx and dy should **never** be in the denominator.
3. Integrate both sides
   - Once we integrate, we only need to add "+C" to one side of the equation
4. If possible, get the answer into y=f(x) form
5. If you are given an initial condition, solve for "+C" by pluggin' in the given x and y value

Note: steps 4 and 5 may be interchanged sometimes

Example 1 Solve each differential equation.

A. $y' = \frac{2x}{y}$

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C$$

$$y^2 = 2x^2 + C$$

$$y = \pm\sqrt{2x^2 + C}$$

B. $y' = x^2 y$

$$\frac{dy}{dx} = x^2 y$$

$$\int \frac{dy}{y} = \int x^2\,dx$$

$$\ln|y| = \frac{1}{3}x^3 + C$$

$$e^{\ln|y|} = e^{\frac{1}{3}x^3 + C}$$

$$y = e^{\frac{1}{3}x^3} \cdot e^C$$

$$\boxed{y = Ce^{\frac{1}{3}x^3}}$$

C. $(x^2+1)\frac{dy}{dx} = xy$

$$\int \frac{dy}{y} = \int \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$\frac{du}{dx} = 2x$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2}\ln|u| + C$

$$\ln|y| = \frac{1}{2}\ln(x^2+1) + C$$

$$\ln|y| = \ln\sqrt{x^2+1} + C$$

$$e^{\ln|y|} = e^{\ln\sqrt{x^2+1} + C}$$

$$y = e^{\ln\sqrt{x^2+1}} \cdot e^C$$

$$\boxed{y = C\sqrt{x^2+1}}$$

D. $\frac{dy}{dx} = y^2 \sin x$

$$\int \frac{dy}{y^2} = \int \sin x\,dx$$

$$-1y^{-1} = -\cos x + C$$

$$\frac{-1}{y} = -\cos x + C$$

$$\frac{1}{y} = \cos x + C$$

$$y = \frac{1}{\cos x + C}$$

Growth and Decay Models

"the rate of change of y is proportional to y" means $\frac{dy}{dt} = ky$ ← constant of proportionality

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$e^{\ln|y|} = e^{kt+C}$$

$$y = e^{kt} \cdot e^{C} \qquad y = Ce^{kt}$$

$k > 0$ growth
$k < 0$ decay

"the rate of change of y is inversely proportional to y" means $\frac{dy}{dt} = \frac{k}{y}$

Example 2 The rate of change of y is proportional to y. When t = 0, y = 2. When t = 2, y = 4. Find the value of y when t = 3.

$(0, 2) \qquad (2, 4)$

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$y = Ce^{kt}$$

$$2 = Ce^{k(0)}$$

$$2 = C$$

$$y = 2e^{kt}$$

$$4 = 2e^{k(2)}$$

$$2 = e^{2k}$$

$$\ln 2 = \ln e^{2k}$$

$$\ln 2 = 2k \cdot \underbrace{\ln e}_{1}$$

$$k = \frac{\ln 2}{2}$$

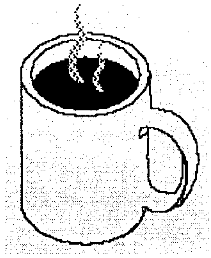
$$y = 2e^{\frac{\ln 2}{2}t}$$

$$\boxed{y = 2e^{\frac{\ln 2}{2}(3)} = 5.657}$$

A connection to precalculus . . .

<u>**Newton's Law of Cooling**</u>—the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium

$$\frac{dT_{obj}}{dt} = K\left(T_{obj} - T_{med}\right)$$



<u>**Example 3**</u> Let y represent the temperature ($F°$) of an object in a room whose temperature is kept at a constant $60°$. If the object cools from $100°$ to $90°$ in 10 minutes, how much longer will it take for its temperature to decrease to $80°$?

$y = T_{obj}$ $\quad T_{med} = 60°$ $\quad (0, 100)$ $\quad (10, 90°)$

$$\frac{dy}{dt} = k(y-60)$$

$$\int \frac{dy}{y-60} = \int k\,dt$$

$$\ln|y-60| = kt + C$$

$e \qquad e$

$$y - 60 = e^{kt} \cdot e^{C}$$

$$y - 60 = Ce^{kt}$$

$$y = 60 + Ce^{kt}$$

$$100 = 60 + Ce^{k(0)}$$

$$40 = Ce^{0}$$

$$40 = C$$

$$y = 60 + 40e^{kt}$$

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$\frac{3}{4} = e^{10k}$$

$$\ln\frac{3}{4} = \ln e^{10k}$$

$$\ln\frac{3}{4} = 10k \cdot \underbrace{\ln e}_{1}$$

$$k = \frac{\ln\frac{3}{4}}{10}$$

$$y = 60 + 40e^{\frac{\ln\frac{3}{4}}{10}t}$$

$$80 = 60 + 40e^{\frac{\ln 3/4}{10}t}$$

$t = 24.094$ mins

$\boxed{14.094 \text{ mins}}$

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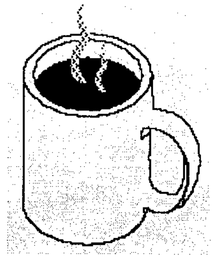

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