

Notes---Differential Equations & Slope Fields

Differential Equations:

- An equation that contains an unknown function and some of its derivatives.
- Its solution has the form $y = f(x)$.
- The general solution has C in it.
- In the particular solution, find C using the initial conditions.

Ex 1) Given $xy' - 3y = 0$ and $y = Cx^3$ is a general solution. If $y = 2$ when $x = -3$, find the particular solution.

$$y = Cx^3$$

$$2 = C(-3)^3$$

$$2 = C(-27) \quad C = -\frac{2}{27}$$

$$y = -\frac{2}{27}x^3$$

Ex 2) Find a general solution for each differential equation.

a) $\frac{dy}{dx} = \frac{e^x}{e^x+1}$

$$\int dy = \int \frac{e^x}{e^x+1} dx$$

$$y = \ln|e^x+1| + C$$

$$y = \ln(e^x+1) + C$$

$$u = e^x + 1$$

$$\frac{du}{dx} = e^x$$

$$du = e^x dx$$

$$\int \frac{du}{u}$$

$$\ln|u| + C$$

b) $\frac{dy}{dx} = x \cos x^2$

$$\int dy = \int x \cos x^2 dx$$

$$y = \frac{1}{2} \sin x^2 + C$$

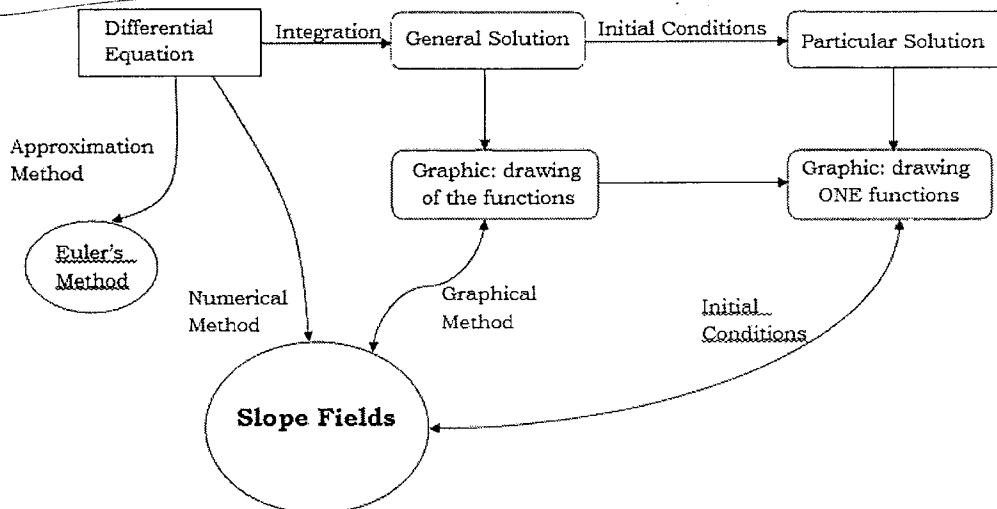
$$u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{1}{2} du = x dx$$

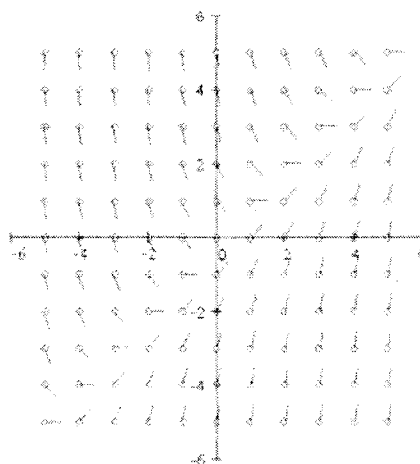
$$\frac{1}{2} \int \cos u du$$

$$\frac{1}{2} \sin u + C$$



Slope Fields

Since the differential equation gives the slope at any point (x, y) , we can use that information to draw a small piece of the linearization at that point. Repeating that process at many points yields an approximation called a slope field, a visual representation of the solution of the differential equation

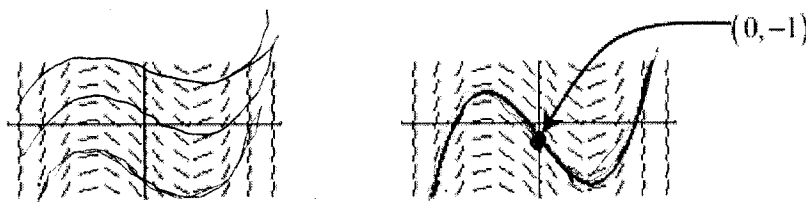


TIPS FOR READING AND/OR DRAWING SLOPE FIELDS

- 1) look for points where the derivative is zero
- 2) look for points where the derivative is undefined
- 3) look for points where the derivative is positive
- 4) look for points where the derivative is negative
- 5) remember that derivative = slope
- 6) think about where the original function is defined (i.e. its domain)
- 7) substitute values of x and y to determine the slope at that particular point (you can use the differential equation if it is given)
- 8) remember that a slope field is a sketch - it is not exact
- 9) remember that a slope field represents the family of solutions to a differential equation (i.e. the family of antiderivatives)
- 10) look for patterns
- 11) when asked to sketch a specific solution passing through a given point, the graph should be close but does not have to be exact

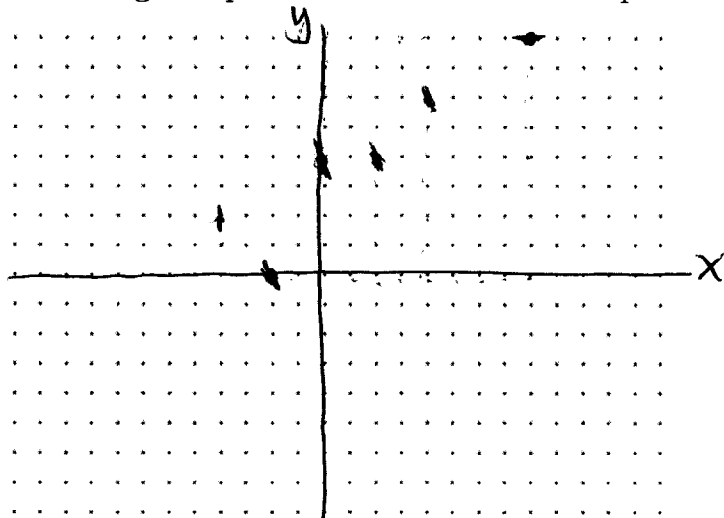
What Do Slope Fields Show You???

The solution curves are hiding in the slope field. Given one point of the particular solution curve, you can sketch the graph from that point, in both directions, to see the graph of the solution. The initial value problem $\frac{dy}{dx} = 3x^2 - 4$ with $y = -1$ when $x = 0$ is shown. Notice how the graph flows through the slope field going both left and right from the starting point $(0, -1)$.



Ex 3) Given $\frac{dy}{dx} = x - y$, find the slopes at the given points and sketch the slope field.

x	y	$\frac{dy}{dx}$
-4	2	-6
-2	0	-2
0	4	-4
2	4	-2
4	6	-2
8	8	0



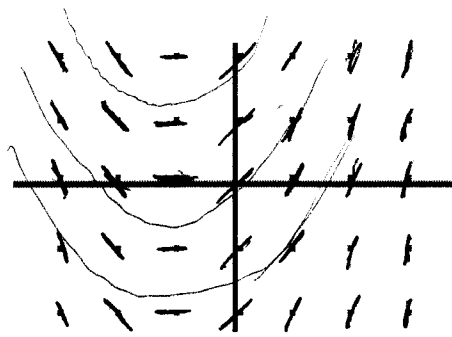
Ex 4) Construct a slope field for the differential equation given below, and solve for the general solution

$$\frac{dy}{dx} = x + 1$$

$$\int dy = \int (x+1) dx$$

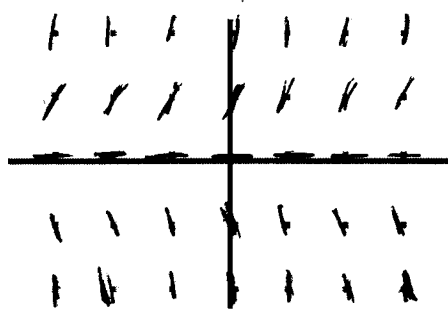
$$y = \frac{1}{2}x^2 + x + C$$

x	$\frac{dy}{dx}$
-3	-2
-2	-1
-1	0
0	1
1	2
2	3
3	4

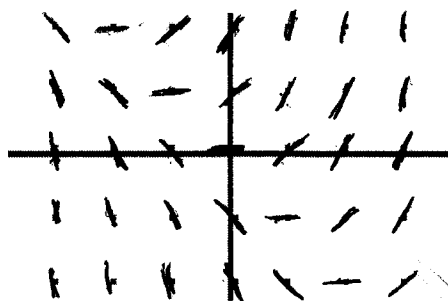


Ex 5) Construct a slope field for the differential equation: $\frac{dy}{dx} = 2y$

y	$\frac{dy}{dx}$
-2	-4
-1	-2
0	0
1	2
2	4



Ex 6) Construct a slope field for the differential equation: $\frac{dy}{dx} = x + y$



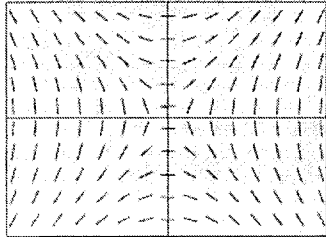
Ex 7) Use slope analysis to match each of the following differential equations with one of the slope fields (a) through (d). (No Calculator)

1. $\frac{dy}{dx} = x - y$ B

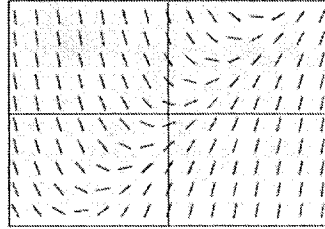
2. $\frac{dy}{dx} = xy$ D

3. ~~A~~ $\frac{dy}{dx} = \frac{x}{y}$
 $x=0 \Rightarrow \frac{dy}{dx} = 0$
 $y=0 \Rightarrow \frac{dy}{dx} = \text{undef}$
 pos slope 1st qtr
 neg slope 2nd qtr

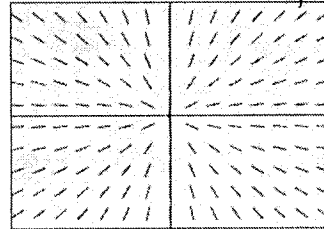
C $\frac{dy}{dx} = \frac{y}{x}$
 $y=0 \Rightarrow \frac{dy}{dx} = 0$
 $x=0 \Rightarrow \frac{dy}{dx} = \text{undef}$



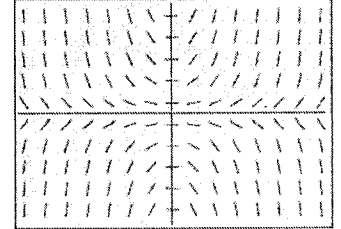
(a)



(b)



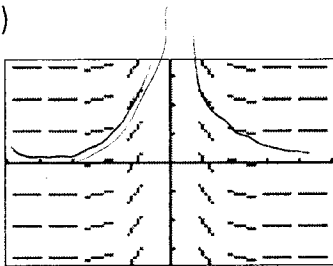
(c)



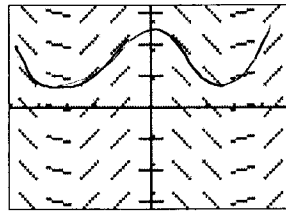
(d)

Ex 8) Match each slope field with the equation that the slope field could represent.

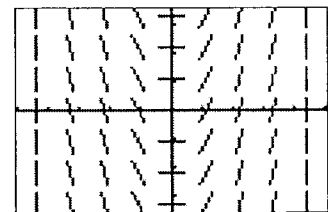
A)



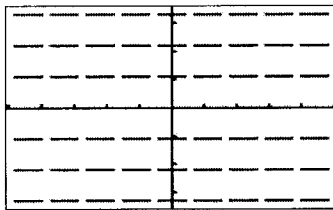
~~B)~~



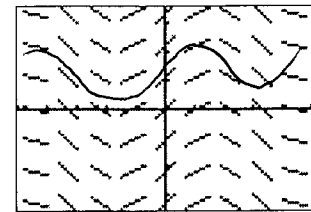
~~C)~~



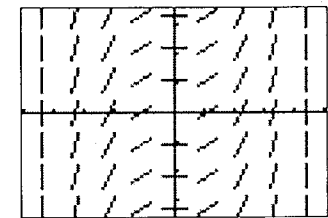
~~D)~~



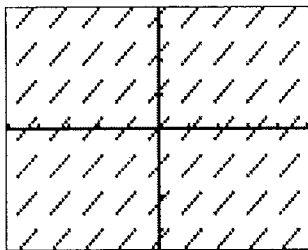
~~E)~~



~~F)~~



~~G)~~



7. $y = 1$ D

8. $y = x$ G

9. $y = x^2$ C

10. $y = \frac{1}{6}x^3$ F

11. $y = \frac{1}{x^2}$ A

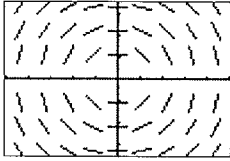
12. $y = \sin x$ E

13. $y = \cos x$ B

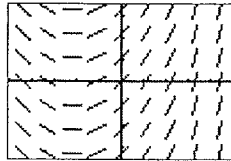
14. $y = \ln|x|$ none

You Try ☺ : Match the Slope Fields with the Differential Equations.

(A)



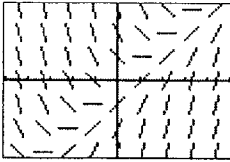
(B)



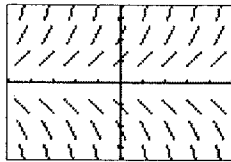
$$\frac{dy}{dx} = \frac{1}{2}x + 1 \quad B$$

$$\frac{dy}{dx} = y \quad D$$

(C)



(D)



$$\frac{dy}{dx} = x - y \quad C$$

$$\frac{dy}{dx} = -\frac{x}{y} \quad A$$

Euler's Method is an iterative process which gives us a numerical method to approximate the particular solution to a differential equation.

1. Begin at the point (x, y) specified by the initial condition.
2. Use the differential equation to find the slope dy/dx at the point.
3. Increase x by a small amount Δx (typically given). Then increase y by a small amount $\Delta y = (dy/dx) \Delta x$. This defines a new point $(x + \Delta x, y + \Delta y)$ which lies along the linearization.
4. Using this new point, return to step 2. Repeating the process constructs a graph to the right of the initial point.
5. To construct a graph moving to the left from the initial point, repeat the process using negative values for Δx .

$$y - y_1 = m(x - x_1) \quad y = y_1 + m(x - x_1)$$

\uparrow new y \uparrow old y \uparrow dy/dx \uparrow Δx Step Size

Ex 9) Given $\frac{dy}{dx} = x + y$ and $f(0) = 0$. Estimate y if $x = 0.5$ using a step size of 0.1.

x	y	dy/dx	
0	0	0	$0 + 0(.1) = 0$
0.1	0	.1	$0 + .1(.1) = .01$
0.2	.01	.21	$.01 + .21(.1) = .031$
0.3	.031	.331	$.031 + .331(.1) = .0641$
0.4	.0641	.4641	$.0641 + .4641(.1) = .11051$
0.5	.11051		

$f(.5) \approx .11051$

Ex 10) Given $\frac{dy}{dx} = \frac{x}{2} + \frac{y}{5}$ and $f(2) = 0$. Estimate $f(3)$ if $\Delta x = 0.5$.

x	y	dy/dx	
2	0	1	$0 + 1(.5) = .5$
2.5	.5	1.35	$.5 + 1.35(.5) = 1.175$
3	1.175		

$f(3) \approx 1.175$