1978 AB5/BC1
Solution

(a) Implicit differentiation gives

$$2x - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 2x$$

$$y' = \frac{y - 2x}{2y - x}$$

(b) There is a vertical tangent when $2y - x = 0$, so $x = 2y$. Substituting into the equation of the curve gives $(2y)^2 - (2y)y + y^2 = 9$, or $3y^2 = 9$. Therefore $y = \pm\sqrt{3}$ and the two points on the curve where the tangents are vertical are $(2\sqrt{3}, \sqrt{3})$ and $(-2\sqrt{3}, -\sqrt{3})$.

(c) $y'' = \dfrac{(2y - x)(y' - 2) - (y - 2x)(2y' - 1)}{(2y - x)^2}$

At the point $(0,3)$, $y' = \dfrac{3-0}{6-0} = \dfrac{1}{2}$ and so $y'' = \dfrac{(6-0)\left(\frac{1}{2} - 2\right) - (3-0)(1-1)}{(6-0)^2} = -\dfrac{1}{4}$

Alternatively, one can use implicit differentiation a second time to get

$$2 - xy'' - y' - y' + 2yy' + 2(y')^2 = 0$$

Substituting $x = 0$, $y = 3$, and $y' = \dfrac{1}{2}$ gives

$$2 - 0 - \frac{1}{2} - \frac{1}{2} + 6y'' + 2\left(\frac{1}{4}\right) = 0 \Rightarrow 6y'' = -\frac{3}{2} \Rightarrow y'' = -\frac{1}{4}$$

1982 AB4
Solution

(a) $x^2 + y^2 = 15^2$

Implicit: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$9 \cdot \frac{1}{2} + 12\frac{dy}{dt} = 0$

$\frac{dy}{dt} = -\frac{3}{8}$

(b) $A = \frac{1}{2}xy$

Implicit: $\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$

$\frac{dA}{dt} = \frac{1}{2}\left(9 \cdot \left(-\frac{3}{8}\right) + 12 \cdot \frac{1}{2}\right)$

$\frac{dA}{dt} = \frac{21}{16}$

1984 AB5
Solution

(a) $A = \pi r^2$

When $r = 3$, $\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi \cdot 3 \cdot \frac{1}{2} = 3\pi$

(b) $V = \frac{1}{3}\pi r^2 h$

$\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$

$28\pi = \frac{1}{3}\pi(9)\frac{dh}{dt} + \frac{2}{3}\pi(3)(4)\left(\frac{1}{2}\right)$

$\frac{dh}{dt} = 8$

or

$V = \frac{1}{3}Ah$

$\frac{dV}{dt} = \frac{1}{3}A\frac{dh}{dt} + \frac{1}{3}h\frac{dA}{dt}$

$28\pi = \frac{1}{3}(9\pi)\frac{dh}{dt} + \frac{1}{3}4(3\pi)$

$\frac{dh}{dt} = 8$

(c) $\frac{dA}{dh} = \frac{\frac{dA}{dt}}{\frac{dh}{dt}} = \frac{3\pi}{8}$

or

$A = \pi r^2$

$\frac{dA}{dh} = 2\pi r \frac{dr}{dh}$

$\frac{dr}{dh} = \frac{\frac{dr}{dt}}{\frac{dh}{dt}} = \frac{1/2}{8} = \frac{1}{16}$

Therefore $\frac{dA}{dh} = 2\pi(3)\left(\frac{1}{16}\right) = \frac{3\pi}{8}$

1985 AB5/BC2
Solution

(a) $V = \pi r^2 h + \frac{4}{3}\pi r^3$

$144\pi = \pi(3)^2 h + \frac{4}{3}\pi(3)^3$

$h = 12$

At this instant, the height is 12 centimeters.

(b) $\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt} + 4\pi r^2 \frac{dr}{dt}$

$261\pi = \pi(3)^2 \frac{dh}{dt} + 2\pi(3)(12)(2) + 4\pi(3)^2(2)$

$\frac{dh}{dt} = 5$

At this instant, the height is increasing at the rate of 5 centimeters per minute.

1992 AB4/BC1
Solution

(a) $\frac{dy}{dx} - \sin y \frac{dy}{dx} = 1$

$\frac{dy}{dx}(1 - \sin y) = 1$

$\frac{dy}{dx} = \frac{1}{1 - \sin y}$

(b) $\frac{dy}{dx}$ undefined when $\sin y = 1$

$y = \frac{\pi}{2}$

$\frac{\pi}{2} + 0 = x + 1$

$x = \frac{\pi}{2} - 1$

(c) $\frac{d^2y}{dx^2} = \frac{d\left(\frac{1}{1 - \sin y}\right)}{dx}$

$= \frac{-\left(-\cos y \frac{dy}{dx}\right)}{(1 - \sin y)^2}$

$= \frac{\cos y\left(\frac{1}{1 - \sin y}\right)}{(1 - \sin y)^2}$

$= \frac{\cos y}{(1 - \sin y)^3}$