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AP Calculus BC

Notes 2.2 : n^{th} term test, geometric series test & telescoping series

Sequence: $a_1, a_2, a_3, a_4, \ldots, a_n$

Series: $a_1 + a_2 + a_3 + a_4 + \ldots + a_n = \sum_{n=1}^{\infty} a_n$

Partial Sums

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

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$S_n = a_1 + a_2 + a_3 + \ldots + a_n$

Properties of Infinite Series

$\sum a_n = A \qquad \sum b_n = B$

1) $\sum c \cdot a_n = c \sum a_n = cA$

2) $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$
$= A \pm B$

Convergent/Divergent Series

In general, if the sequence of the partial sum, $\{S_n\}$, converges to "S", then the series,

$\sum_{n=1}^{\infty} a_n$, converges.

If $\{S_n\}$, diverges, then the series diverges.

Ex1) Determine the convergence of the series.

A) $\sum_{n=1}^{\infty} \frac{1}{2^n}$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$

$S_4 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

$\lim_{n \to \infty} S_n = 1$

$\boxed{\text{Series converges}}$

B) $\sum_{n=1}^{\infty} n$

$S_1 = 1$

$S_2 = 1 + 2 = 3$

$S_3 = 1 + 2 + 3 = 6$

$\lim_{n \to \infty} S_n = \text{DNE}$ (∞)

$\boxed{\text{Series diverges}}$

C) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$ telescoping series.

$S_1 = \left(1 - \frac{1}{2}\right) = \frac{1}{2}$

$S_2 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3}$

$S_3 = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{3}{4}$

$S_4 = \frac{4}{5}$

$S_5 = \frac{5}{6}$

$\lim_{n \to \infty} S_n = 1$ $\boxed{\text{Series converges}}$

GEOMETRIC SERIES TEST

Geometric Series have a common ratio.

$\sum_{n=1}^{\infty} a(r)^{n-1}$ or $\sum_{n=0}^{\infty} a(r)^{n}$

$a + ar^1 + ar^2 + \cdots$

If $|r| < 1$, the series converges

If $|r| \geq 1$, the series diverges.

Sum of a Convergent Geometric Series

$$\boxed{\text{sum} = \frac{a_1}{1-r}}$$

$a_1 \leftarrow$ 1st term

$r \leftarrow$ common ratio

Ex 2) Determine the convergence of the series.

A) $10 + 5 + \frac{5}{2} + \frac{5}{4} \cdots$

geom. series $r = \frac{1}{2}$ $\quad \left|\frac{1}{2}\right| < 1$

Series converges by the geom series test

$\text{sum} = \frac{10}{1-\frac{1}{2}} = 20$

B) $-3 - 6 - 12 - 24 - \cdots$

geom. series $r = 2$ $\quad |2| > 1$

series diverges by the geom. series test

C) $\sum_{n=0}^{\infty} \frac{1}{9}\left(\frac{1}{3}\right)^n = \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \ldots$

geom. series $r = \frac{1}{3}$ $\quad \left|\frac{1}{3}\right| < 1$

series converges by the geom. series test

$\text{sum} = \frac{\frac{1}{9}}{1-\frac{1}{3}} = \frac{\frac{1}{9}}{\frac{2}{3}} = \frac{1}{6}$

D) $\sum_{n=0}^{\infty} (-4)^n = 1 + {-4} + 16 + {-64} + 256 + \ldots$

geom. series $r = -4$ $\quad |-4| > 1$

series diverges by the geom. series test

E) Look back to example 1A.

$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$\left(\frac{1}{2}\right)^n$

geom. series $r = \frac{1}{2}$ $\quad \left|\frac{1}{2}\right| < 1$

series converges by the geom. series test

$\text{sum} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

n^{th} Term Test (divergence)

Limit of the n^{th} term of a divergent series

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ex 3) Apply the n^{th} term test for divergence.

A) $\sum_{n=1}^{\infty} n^2$ $\quad \lim_{n\to\infty} n^2 = \infty \leftarrow \neq 0$ (DNE)

series diverges by the n^{th} term test

B) $\sum_{n=1}^{\infty} \frac{n+1}{n}$ $\quad \lim_{n\to\infty} \frac{n+1}{n} = 1 \neq 0$

series diverges by the n^{th} term test

C) $\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \lim_{n\to\infty} \frac{1}{n} = 0$ $\quad n^{th}$ term test fails

$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ harmonic series diverges

D) $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ $\quad \lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \neq 0$

series diverges by the n^{th} term test