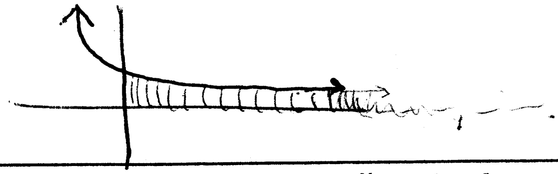
AP Calculus BC

Notes: Improper Integrals



Consider the infinite region that lies under the curve $y = e^{-x}$ in the first quadrant. If you were asked to find this area, what would you say? We will see that this area is actually finite.

There are 2 types of improper integrals.

Type #1:

- At least one of the limits of integration is infinite
- Interval is not bounded
- To evaluate these types of improper integrals we will use the following formulas:

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_a^b f(x)dx$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

Where c is any real number.

Important!! Learn and Memorize!!

In formulas 1 & 2: If the limit is finite, the improper integral ***converges*** and the limit is the value of the improper integral. If the limit fails to exist (is infinite), the improper integral ***diverges***.

In formula 3: Both integrals must converge (have finite limits) in order for the improper integral to converge. Otherwise, the integral diverges.

10

1) $\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}dx = \lim_{b\to\infty} \left[-1x^{-1} + c \Big|_1^b\right]$ $\to x^{-2}$

$= \lim_{b\to\infty} \left[\frac{-1}{b} + c - (-1+c)\right] = \lim_{b\to\infty}\left[-\frac{1}{b}+1\right]$

$0 + 1 = 1$

$\boxed{\text{converges}}$

L I A T E

Ex 2) $\int_{-\infty}^0 xe^{-2x}dx = \lim_{a\to-\infty}\int_a^0 xe^{-2x}dx =$

	u	dv
$+$	x	e^{-2x}
$-$	1	$-\frac{1}{2}e^{-2x}$
$+$	0	$\frac{1}{4}e^{-2x}$

$= \lim_{a\to-\infty}\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c \Big|_a^0\right]$

$= \lim_{a\to-\infty}\left[-\frac{1}{4} + c - \left(-\frac{1}{2}ae^{-2a} - \frac{1}{4}e^{-2a} + c\right)\right]$

$= -\frac{1}{4} + \frac{1}{2}\cdot -\infty + \frac{1}{4}\cdot\infty = \boxed{\text{diverges}}$

Ex 3) $\int_0^\infty e^{-x}dx = \lim_{b\to\infty}\int_0^b e^{-x}dx$

$u = -x$

$\frac{du}{dx} = -1 \quad -du = dx$

$-\int e^u du = -e^u + c$

$= \lim_{b\to\infty}\left[-e^{-x} + c\right]\Big|_0^b$

$= \lim_{b\to\infty}\left[\frac{-1}{e^b} - (-1)\right] = 0 + 1 = 1 \ \boxed{\text{converges}}$

||

Ex 4) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$

$$= \lim_{a\to-\infty} \int_{a}^{0} \frac{1}{1+x^2}\,dx + \lim_{b\to\infty} \int_{0}^{b} \frac{1}{1+x^2}\,dx$$

$$= \lim_{a\to-\infty} \left[\arctan x + C \Big|_{a}^{0}\right] + \lim_{b\to\infty} \left[\arctan x \Big|_{0}^{b}\right]$$

$$= \lim_{a\to-\infty} \left[\underbrace{\arctan 0}_{0} - \arctan a\right] + \lim_{b\to\infty} \left[\arctan b - \arctan 0\right]$$

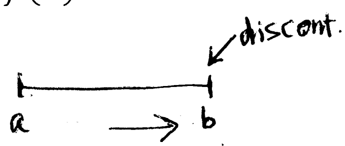
$$= 0 - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi \quad \boxed{\text{converges}}$$

Type #2: $\int_a^b f(x)dx$ where $f(x)$ has a point of discontinuity.

- Finite limits of integration
- Function is not bounded
- Function is discontinuous

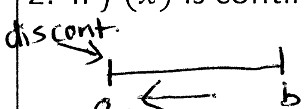
- To evaluate these types of improper integrals we will use the following formulas:

1. If $f(x)$ is continuous on $[a, b)$ and has infinite discontinuity at b, then



$$\int_a^b f(x)dx = \lim_{c\to b^-} \int_a^c f(x)dx$$

2. If $f(x)$ is continuous on $(a, b]$, and has infinite discontinuity at a, then



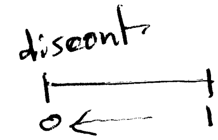
$$\int_a^b f(x)dx = \lim_{c\to a^+} \int_c^b f(x)dx$$

3. If $f(x)$ is continuous on $[a, c) \cup (c, b]$ at which f has infinite discontinuity at c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

The same rules of convergence and divergence as before apply to these formulas.

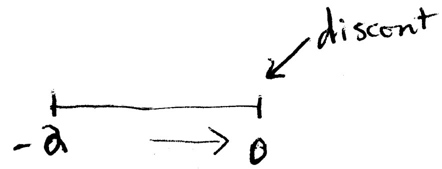
Ex 5) $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ discont at $x=0$



$$\lim_{c\to 0^+} \int_c^1 \frac{dx}{\sqrt[3]{x}} \quad \to x^{-1/3} \qquad = \lim_{c\to 0^+} \left[\frac{3}{2} x^{2/3} + C \Big|_c^1 \right]$$

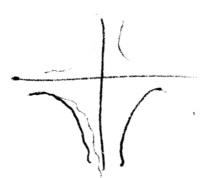
$$= \lim_{c\to 0^+} \left[\frac{3}{2} - \frac{3}{2} c^{2/3} \right] = \frac{3}{2} - 0 = \frac{3}{2} \quad \boxed{\text{converges}}$$

Ex 6) $\int_{-2}^0 \frac{1}{x^3}\,dx$ discont. at $x=0$

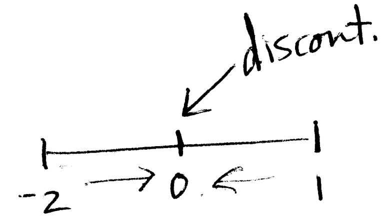


$$\lim_{c\to 0^-} \int_{-2}^{c} \frac{1}{x^3}\,dx \quad \to x^{-3} \qquad = \lim_{c\to 0^-} \left[-\frac{1}{2} x^{-2} + C \Big|_{-2}^{c} \right]$$

$$= \lim_{c\to 0^-} \left[\frac{-1}{2c^2} - \left(-\frac{1}{8}\right) \right] = -\infty + \frac{1}{8} \quad \boxed{\text{diverges}}$$



Ex 7) $\int_{-2}^1 \frac{dx}{x^3}$ discont. at $x=0$



$$\int_{-2}^{0} \frac{dx}{x^3} + \int_0^1 \frac{dx}{x^3}$$

$$\underbrace{\lim_{c\to 0^-} \int_{-2}^{c} \frac{dx}{x^3}}_{\text{see \#6}} + \lim_{c\to 0^+} \int_c^1 \frac{dx}{x^3} = \boxed{\text{diverges}}$$