The Integral and p-series Tests

* tests for series that have positive terms

Integral Test

conditions: 1) f must be positive, continuous, and decreasing

2) $f(n) = a_n$

Then $\sum a_n$ and $\int_{1^*}^{\infty} f(x)\,dx$ will either both diverge or both converge

*could be another #

EX1 Determine convergence.

A. $\sum \frac{n}{n^2+1}$ $\quad f(n) = \frac{n}{n^2+1}$ pos ✓ cont. ✓ decr ✓

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx$$

$$\lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx$$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2}\ln|u| + c$

$$\lim_{b\to\infty} \left[\frac{1}{2}\ln|x^2+1| + c \Big|_1^b \right]$$

$$\lim_{b\to\infty} \left[\underbrace{\frac{1}{2}\ln(b^2+1)}_{\infty} - \frac{1}{2}\ln(2) \right]$$

$$\infty - \frac{1}{2}\ln 2 = \infty$$

integral diverges ⇒ series diverges by the Integral Test

B. $\sum \frac{n}{e^n}$ $f(n) = \frac{n}{e^n}$ pos ✓ cont ✓ decr ✓

$$\int_1^\infty \frac{x}{e^x}\,dx = \lim_{b\to\infty} \int_1^b x \cdot e^{-x}\,dx$$

u	dv
$+x$	e^{-x}
-1	$-e^{-x}$
$+0$	e^{-x}

$$= \lim_{b\to\infty} \left[-xe^{-x} - e^{-x} + C \Big|_1^b \right]$$

$$= \lim_{b\to\infty} \left[-\frac{b}{e^b} - \frac{1}{e^b} - \left(-1 \cdot e^{-1} - e^{-1}\right) \right]$$

$$= 0 - 0 + \frac{1}{e} + \frac{1}{e} = \frac{2}{e}$$

integral converges $\Rightarrow$ series converges by the Integral test

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p-series has form $\sum_{n=r}^{\infty} \frac{1}{n^p}$ (p is a constant)

terms: $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

p-series Test

If $p > 1$, the series converges.
If $p \leq 1$, the series diverges.

EX2 Determine convergence.

A. $\sum \frac{1}{n \cdot \sqrt[3]{n}} = \sum \frac{1}{n^1 \cdot n^{1/3}} = \sum \frac{1}{n^{4/3}}$

p-series $p = 4/3$ $\quad \frac{4}{3} > 1$ series converges by the p-series test

B. $\frac{1}{1} + \frac{1}{8} + \frac{1}{27} + \dots = \sum_{n=1}^{\infty} \frac{1}{n^3}$

p-series $p = 3$ $\quad 3 > 1$ series converges by the p-series test

C. $\frac{4}{\sqrt[3]{1}} + \frac{4}{\sqrt[3]{2}} + \frac{4}{\sqrt[3]{3}} + \dots \sum_{n=1}^{\infty} \frac{4}{\sqrt[3]{n}} = 4 \sum \frac{1}{n^{1/3}}$

p-series $p = \frac{1}{3}$ $\quad \frac{1}{3} < 1$ series diverges by the p-series test

$\sum \frac{1}{n^2}$ p-series $p = 2$

$\sum \frac{1}{n^2+1}$ NOT a p-series

order

1. nth term test for div.
2. special series?
   - p-series
   - geometric series
   - telescoping series
3. integral test