Notes: Inverse Trig Functions, Derivatives & Integration

Function	Domain	Range
arcsin x	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$
arccos x	$-1 \le x \le 1$	$0 \le y \le \pi$
arctan x	$(-\infty, \infty)$	$-\pi/2 < y < \pi/2$
arccsc x	$(-\infty, -1] \cup [1, \infty)$	$-\pi/2 \le y \le \pi/2 \quad y \ne 0$
arcsec x	$(-\infty, -1] \cup [1, \infty)$	$0 \le y \le \pi, \ y \ne \pi/2$
arccot x	$(-\infty, \infty)$	$0 < y < \pi$

Evaluate:

a. $sin^{-1}(0) = 0$

b. $tan^{-1}(1) = \pi/4$

c. $tan^{-1}(-1) = -\pi/4$

d. $cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

e. $sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\pi/4$

f. $cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

g. $arccot(1) = \frac{\pi}{4}$

h. $\arccos(-0.628) = 2.250$ (calc)

Six basic trig functions

Three basic trig functions:

$\sin\theta = \frac{\text{Opp}}{\text{Hyp}}$

$\cos\theta = \frac{\text{Adj}}{\text{Hyp}}$

$\tan\theta = \frac{\text{Opp}}{\text{Adj}}$

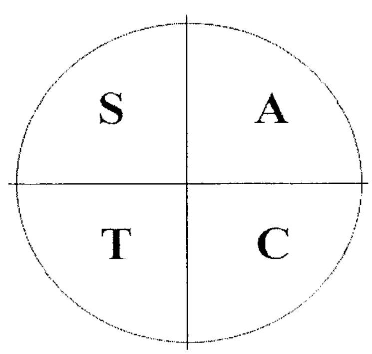
Reciprocal trig functions:

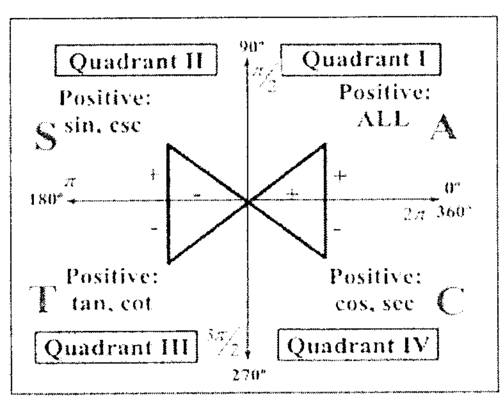
$\csc\theta = \frac{\text{Hyp}}{\text{Opp}}$

$\sec\theta = \frac{\text{Hyp}}{\text{Adj}}$

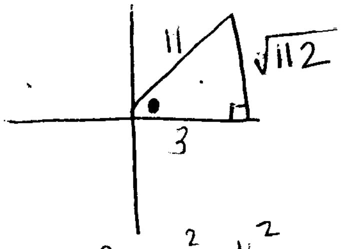
$\cot\theta = \frac{\text{Adj}}{\text{Opp}}$

(sine-opp./hyp;cosine-adj./hyp;tangent-opp./adj.)





Evaluate:



a. $\sin\left(\arccos\frac{3}{11}\right)$ a h

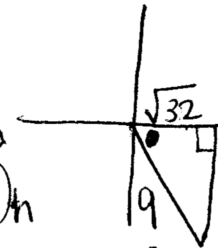
$\boxed{\frac{\sqrt{112}}{11}}$

$3^2 + y^2 = 11^2$

$y^2 = 112$

$y = \sqrt{112}$

b. $\tan\left(\arcsin -\frac{7}{9}\right)$ o h



$\boxed{\frac{-7}{\sqrt{32}}}$

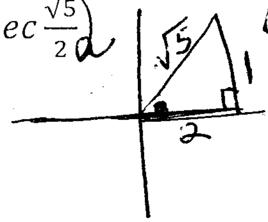
$x^2 + (-7)^2 = 9^2$

$x^2 + 49 = 81$

$x^2 = 32$

$x = \sqrt{32}$

c. $\tan\left(\arcsec\frac{\sqrt{5}}{2}\right)$ h a



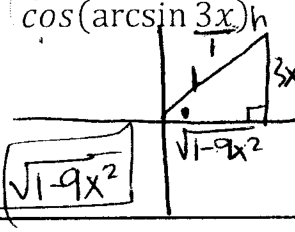
$\boxed{\frac{1}{2}}$

$2^2 + y^2 = \sqrt{5}^2$

$y^2 = 1$

$y = 1$

d. $\cos(\arcsin 3x)$ o h



$\boxed{\sqrt{1-9x^2}}$

$m^2 + (3x)^2 = 1^2$

$m^2 = 1 - 9x^2$

$m = \sqrt{1-9x^2}$

Derivatives of Inverse Trig Functions

Let u be a differentiable function of x.

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Find $\frac{dy}{dx}$.

$y = \sin^{-1}x$

see unit 2 notes

$$y' = \frac{1}{\sqrt{1-x^2}}$$

Find each derivative.

a. $y = sin^{-1}(2x)$

$$y' = \frac{2}{\sqrt{1-(2x)^2}} = \frac{2}{\sqrt{1-4x^2}}$$

b. $y = tan^{-1}(3x)$

$$y' = \frac{3}{1+(3x)^2} = \frac{3}{1+9x^2}$$

c. $y = sec^{-1}(e^{2x})$

$$y' = \frac{2\cancel{e^{2x}}}{\cancel{|e^{2x}|}\sqrt{(e^{2x})^2-1}} = \frac{2}{\sqrt{e^{4x}-1}}$$

d. $g(x) = \frac{\arcsin(3x)}{x}$

$$g'(x) = \frac{x \cdot \frac{3}{\sqrt{1-(3x)^2}} - \arcsin(3x) \cdot}{x^2}$$

Integration Involving Inverse Trig Functions

Let u be a differentiable function of x, and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2-u^2}} = arcsin\frac{u}{a} + C$$

$$\int \frac{du}{a^2+u^2} = \frac{1}{a}arctan\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}arcsec\frac{|u|}{a} + C$$

Integrate.

a. $\int \frac{2}{9+4x^2}dx$

$a^2 = 9$ $\quad u^2 = 4x^2$

$a = 3$ $\quad u = 2x$

$\frac{du}{dx} = 2$

$du = 2dx$

$$\int \frac{du}{a^2+u^2}$$

$$= \frac{1}{a}\arctan\frac{u}{a} + C$$

$$= \boxed{\frac{1}{3}\arctan\frac{2x}{3} + C}$$

b. $\int \frac{3x}{\sqrt{1-9x^2}}dx$

$u = 1-9x^2$

$\frac{du}{dx} = -18x$

$-\frac{1}{18}du = x\,dx$

$$-\frac{1}{18} \cdot 3\int \frac{du}{\sqrt{u}}$$

$$-\frac{1}{6}\int u^{-1/2}du$$

$$-\frac{1}{3}u^{1/2} + C$$

$$\boxed{-\frac{1}{3}\sqrt{1-9x^2} + C}$$

c. $\int \frac{x}{2x^2\sqrt{4x^4-36}}dx$

$u^2=4x^4$ $a^2=36$

$u=2x^2$ $a=6$ $\frac{1}{4}\int \frac{du}{u\sqrt{u^2-a^2}}$

$\frac{du}{dx}=4x$ $\frac{1}{4}\left[\frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}\right]+C$

$\frac{1}{4}du=xdx$

$\frac{1}{24}\operatorname{arcsec}\frac{2x^2}{6}+C$

$\boxed{\frac{1}{24}\operatorname{arcsec}\frac{x^2}{3}+C}$

d. $\int_0^1 \frac{1}{\sqrt{4-x^2}}dx$

$a^2=4$ $u^2=x^2$

$a=2$ $u=x$

$\frac{du}{dx}=1$

$du=dx$

$\int\frac{du}{\sqrt{a^2-u^2}}$

$\arcsin\frac{u}{a}+C$

$\arcsin\frac{x}{2}+C\Big|_0^1$

$\arcsin\frac{1}{2}-\arcsin 0$

$\frac{\pi}{6}-0$

$\boxed{\frac{\pi}{6}}$

e. $\int_{\sqrt{3}}^{3}\frac{4}{9+x^2}dx$

$a^2=9$ $u^2=x^2$ $4\int\frac{du}{a^2+u^2}$

$a=3$ $u=x$

$du=dx$ $4\cdot\frac{1}{a}\arctan\frac{u}{a}+C$

$\frac{4}{3}\arctan\frac{x}{3}+C\Big|_{\sqrt{3}}^{3}$

$\frac{4}{3}\arctan 1-\frac{4}{3}\arctan\frac{\sqrt{3}}{3}$

$\frac{4}{3}\cdot\frac{\pi}{4}-\frac{4}{3}\cdot\frac{\pi}{6}$

$\frac{\pi}{3}-\frac{2}{9}\pi=\frac{3\pi-2\pi}{9}=\boxed{\frac{\pi}{9}}$

f. $\int\frac{x^4-1}{x^2+1}dx$

$\int\frac{\cancel{(x^2+1)}(x^2-1)}{\cancel{x^2+1}}dx$

$\int(x^2-1)dx$

$\boxed{\frac{1}{3}x^3-x+C}$

g. $\int\frac{t}{t^4+16}dt$

$u^2=t^4$ $a^2=16$

$u=t^2$ $a=4$

$\frac{du}{dt}=2t$ $\frac{1}{2}\int\frac{du}{u^2+a^2}$

$\frac{1}{2}du=tdt$

$\frac{1}{2}\cdot\frac{1}{a}\arctan\frac{u}{a}+C$

$\boxed{\frac{1}{8}\arctan\frac{t^2}{4}+C}$

h. $\int\frac{dx}{x^2+4x+13}$ complete the sq.

$\left(\frac{4}{2}\right)^2=4$ $\underbrace{x^2+4x+4}+13-4$

$(x+2)^2+9$

$\int\frac{dx}{(x+2)^2+9}$ $\int\frac{du}{u^2+a^2}$

$\frac{1}{a}\arctan\frac{u}{a}+C$

$u^2=(x+2)^2$ $a^2=9$

$u=x+2$ $a=3$

$du=dx$

$\boxed{\frac{1}{3}\arctan\frac{x+2}{3}+C}$