AP Calculus BC Review Notes Area and Volume

You should be able to . . .

- Find the area between 2 functions or the area of a polar graph.
- Find the volume of a solid of revolution using the disk, washer, or shell method.
- Find the volume of a solid in a plane using the cross-section method.
- Find the arc length of a function, parametric curve, or polar graph.

AREA

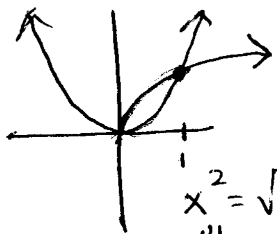
top bottom

$$A=\int_a^b\left(\begin{matrix}\text{upper}\\ \text{function}\end{matrix}\right)-\left(\begin{matrix}\text{lower}\\ \text{function}\end{matrix}\right)dx,\qquad a\le x\le b$$

OR

$$A=\int_c^d\left(\begin{matrix}\text{right}\\ \text{function}\end{matrix}\right)-\left(\begin{matrix}\text{left}\\ \text{function}\end{matrix}\right)dy,\qquad c\le y\le d$$

Example: Determine the area of the region enclosed by $y=x^2$ and $y=\sqrt{x}$.



$x^2=\sqrt{x}$

$x^4=x$

$x^4-x=0$

$x(x^3-1)=0$

$x=0 \quad x^3-1=0$

$x=1$

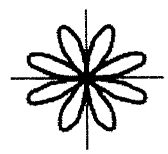
$$\int_0^1(\sqrt{x}-x^2)dx = \frac{2}{3}x^{3/2}-\frac{1}{3}x^3+C\Big|_0^1$$

$$=\frac{2}{3}-\frac{1}{3}-(0-0)$$

$$=\boxed{\frac{1}{3}}$$

Polar Area $\quad A=\int_\alpha^\beta \frac{1}{2}r^2\,d\theta$

Example: Find the area of one petal for r = 2sin 4θ



$2\sin 4\theta=0$

$\sin 4\theta=0$

$4\theta=\sin^{-1}(0)$

$4\theta=0,\pi,2\pi,\text{etc}\quad \theta=0,\frac{\pi}{4},\frac{\pi}{2},\text{etc.}$

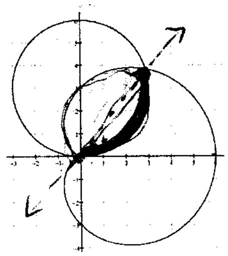
$$\int_0^{\pi/4}\frac{1}{2}(2\sin 4\theta)^2\,d\theta$$

.785

Example: The figure below shows the graphs of $r = 6\sin\theta$ and $r = 3 + 3\cos\theta$ for $0 \le \theta \le 2\pi$.

a) Set up an equation to find the value of θ for the intersection(s) of both graphs. Find the polar coordinates of the point(s) of intersection.

b) Set up an expression with two or more integrals to find the area common to both curves.



a) $6\sin\theta = 3+3\cos\theta$

$\theta = 0.927295522,\ \pi$

(r,θ)

$(0,\pi)\ (4.8, .927)$

b) $\frac{1}{2}\int_{.927}^{\pi}(3+3\cos\theta)^2\,d\theta = \frac{13.33301533}{2} = 6.6665076$

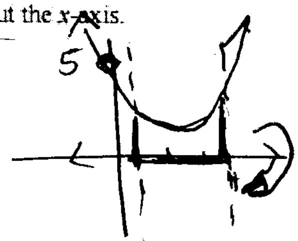
$+$

$\frac{1}{2}\int_0^{.927}(6\sin\theta)^2\,d\theta = 4.025656985$

$\boxed{10.692}$

Volume: Disk Method $V=\int_a^b \pi[f(x)]^2\,dx$ **Volume: Washer Method** $V=\int_a^b \pi\{[f(x)]^2-[g(x)]^2\}\,dx$

Example 1 Determine the volume of the solid obtained by rotating the region bounded by $y=x^2-4x+5$, $x=1$, $x=4$, and the x-axis about the x-axis.



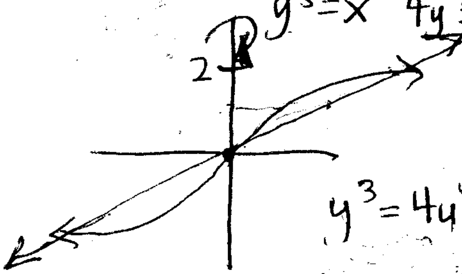
$$\pi\int_1^4 (x^2-4x+5)^2\,dx$$

$\boxed{15.6\pi}$

49.009

Example 2 Determine the volume of the solid obtained by rotating the portion of the region bounded by $y=\sqrt[3]{x}$ and $y=\frac{x}{4}$ that lies in the first quadrant about the y-axis.

$y^3=x$ $4y=x$



$$\pi\int_0^2 (4y)^2-(y^3)^2\,dy = \boxed{\begin{matrix}24.381\pi\\ 76.595\end{matrix}}$$

$y^3=4y$ $y^3-4y=0$

$y(y^2-4)=0$ $y=0,2,-2$

Example 3 Determine the volume of the solid obtained by rotating the region bounded by $y=x^2-2x$ and $y=x$ about the line $y=4$.

$x^2-2x=0$

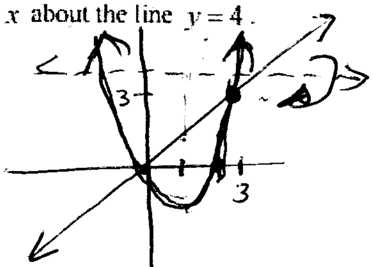
$x(x-2)=0$

$x=0, x=2$

$x^2-2x=x$

$x^2-3x=0$

$x(x-3)=0$ $x=0,3$

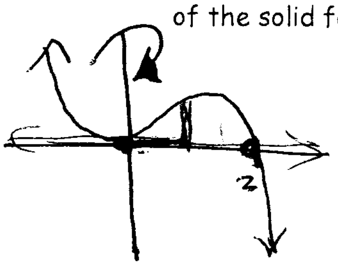


$$\pi\int_0^3 \left[(4-(x^2-2x))^2-(4-x)^2\right]dx$$

$\boxed{\begin{matrix}30.6\pi\\ 96.133\end{matrix}}$

Volume: Shell Method Formula is $V = 2\pi\int_a^b \underbrace{x}_{\text{radius}}\,\underbrace{f(x)}_{\text{height}}\,\underbrace{dx}_{\text{thickness}}$

Example: Given the region in the first quadrant bounded by $y=2x^2-x^3$ and $y=0$. Find the volume of the solid formed when the region is revolved the y-axis.



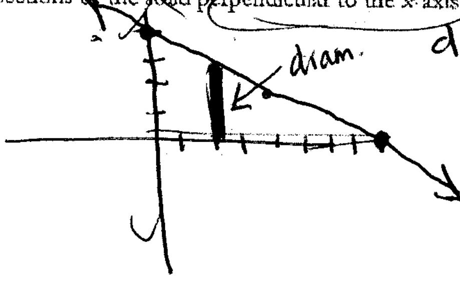
$$2\pi\int_0^2 x\cdot(2x^2-x^3)\,dx = \boxed{\begin{matrix}3.2\pi\\ 10.053\end{matrix}}$$

Volume by Cross-Sections $V=\int_a^b A(x)\,dx$ $V=\int_c^d A(y)\,dy$

Example 4: The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line $x+2y=8$. If the known cross sections of the solid perpendicular to the x-axis are semicircles, what is the volume of the solid?

$2y=-x+8$

$y=-\frac{1}{2}x+4$



dx

$A=\frac{1}{2}\pi r^2$

$$\int_0^8 \frac{1}{2}\pi\left(\frac{-\frac{1}{2}x+4}{2}\right)^2 dx \quad \boxed{16.755}$$

$$\int_0^8 \frac{1}{2}\pi\left(-\frac{1}{4}x+2\right)^2 dx = \boxed{\frac{16\pi}{3}}$$

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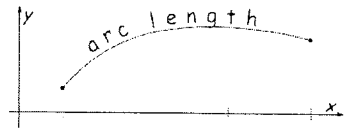
Arc Length

Function	$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$
Parametric equations	$L=\int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$
Polar	$L=\int_a^b \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$



Example: (Calculator Permitted) Which of the following gives the best approximation of the length of the arc of $y=\cos(2x)$ from $x=0$ to $x=\frac{\pi}{4}$?

(A) 0.785 (B) 0.955 (C) 1.0 (D) 1.318 (E) 1.977

$$\int_0^{\pi/4} \sqrt{1+(-2\sin(2x))^2}\,dx$$

Example: (no calculator)

Determine the length of the parametric curve given by the following parametric equations.

$x=3\sin(t)$ $\qquad y=3\cos(t)$ $\qquad 0\le\theta\le 2\pi$

$$\int_0^{2\pi} \sqrt{(3\cos t)^2+(-3\sin t)^2}\,dt = \int_0^{2\pi} \sqrt{9\cos^2 t+9\sin^2 t}\,dt$$

$\sqrt{9}$

$$= \int_0^{2\pi} 3\,dt = 3t + C\Big|_0^{2\pi} = \boxed{6\pi}$$

Example: (no calculator)

Write down an integral expression for the length of the curve $r=\sin\theta+\theta$ for $0\le\theta\le\pi$ but do not compute the integral.

$$\frac{dr}{d\theta} = \cos\theta + 1$$

$$\int_0^{\pi} \sqrt{(\sin\theta+\theta)^2+(\cos\theta+1)^2}\,d\theta$$