AP Calculus BC Review Notes Integration Techniques

You should be able to . . .

- Use basic rules to integrate an indefinite function
- Use advanced techniques (u-substitution, partial fractions, integration by parts) to integrate a function
- Evaluate an improper integral

Some basic integration formulas

$\int k\,du = ku + C, k\ is\ a\ constant$	$\int u^n du = \frac{u^{n+1}}{n+1} + C, n \neq 1$
$\int e^u du = e^u + C$	$\int a^u du = \frac{a^u}{\ln a} + C$
$\int \cos u\, du = \sin u + C$	$\int \sin u\, du = -\cos u + C$
$\int \sec^2 u\, du = \tan u + C$	$\int \csc^2 u\, du = -\cot u + C$
$\int \sec u \tan u\, du = \sec u + C$	$\int \csc u \cot u\, du = -\csc u + C$
$\int \tan u\, du = ln\|\sec u\| + C$	$\int \cot u\, du = -\ln\|\csc u\| + C$
$\int \frac{1}{u^2+a^2} du = \frac{1}{a} tan^{-1}\left(\frac{u}{a}\right) + C$	$\int \frac{1}{\sqrt{a^2-u^2}} du = sin^{-1}\left(\frac{u}{a}\right) + C$
$\int \frac{1}{u^2-a^2} du = \frac{1}{2a} \ln\left\|\frac{u-a}{u+a}\right\| + C$	$\int \frac{1}{\sqrt{u^2 \pm a^2}} du = ln\left\|u + \sqrt{u^2 \pm a^2}\right\| + C$

Ex 1) $\int (5x+4)^5\, dx$

$u = 5x+4$

$\frac{du}{dx} = 5 \rightarrow \frac{1}{5}du = dx$

$\frac{1}{5}\int u^5 du = \frac{1}{5} \cdot \frac{1}{6}u^6 + C$

$= \frac{1}{30}(5x+4)^6 + C$

Ex 2) $\int \frac{\sin x}{\cos^5 x}\, dx$

$u = \cos x$

$\frac{du}{dx} = -\sin x \rightarrow -du = \sin x\, dx$

$-\int \frac{du}{u^5} = -\int u^{-5} du = \frac{1}{4}u^{-4} + C$

$\frac{1}{4\cos^4 x} + C$

Ex 3) $\int x\sqrt{x+3}\, dx$

$u = x+3 \rightarrow x = u-3$

$\frac{du}{dx} = 1 \rightarrow du = dx$

$\int (u-3)\sqrt{u}\, du = \int \left(u^{3/2} - 3u^{1/2}\right) du = \frac{2}{5}u^{5/2} - 2u^{3/2} + C$

$= \frac{2}{5}(x+3)^{5/2} - 2(x+3)^{3/2} + C$

Integration by Parts

Let

$$u = f(x) \qquad dv = g'(x)\,dx$$
$$du = f'(x)\,dx \qquad v = g(x)$$

Then the formula becomes

$$\int u\,dv = uv - \int v\,du.$$

- Choose u, dv in such a way that:
  1. u is easy to *differentiate*.
  2. dv is easy to *integrate*.
  3. $\int v\,du$ is easier to compute that $\int u\,dv$.
- Sometimes it is necessary to integrate by parts more than once.

L - Logarithmic

I - Inverse trigonometric

A - Algebraic

T - Trigonometric

E - Exponential

Tabular Method for Integration by parts

Tabular Method

(1) Differentiate $p(x)$ repeatedly until you obtain 0 and list the results in the first column.
(2) Integrate $f(x)$ repeatedly and list the results in the second column.
(3) Draw an arrow from each entry in the first column to the entry one row down in the second column.
(4) Label the arrows with alternating + and − signs, starting with +.
(5) For each arrow, form the product of the expressions at its tip and tail and multiply by −1 if the arrow is labeled −. Add up these products to obtain the value of the integral.

	u	dV
+	x^2	$\cos x$
−	$2x$	$\sin x$
+	2	$-\cos x$
−	0	$-\sin x$

Ex 4) Find $\int e^x \cos x\,dx$

$u = \cos x \quad dV = e^x dx$

$\frac{du}{dx} = -\sin x \quad V = \int e^x dx = e^x$

$e^x \cos x + \int e^x \sin x\,dx$

$u = \sin x \quad dV = e^x dx$

$\frac{du}{dx} = \cos x \quad V = e^x$

$\sin x\, e^x - \int e^x \cos x\,dx$

$\int e^x \cos x\,dx = e^x \cos x + \sin x \cdot e^x - \int e^x \cos x$

$2\int e^x \cos x\,dx = e^x \cos x + e^x \sin x$

$\int e^x \cos x\,dx = \boxed{\frac{1}{2} e^x (\cos x + \sin x) + C}$

Ex 5) Find $\int x^2 \cos x\,dx$ $\quad \boxed{x^2 \sin x + 2x\cos x - 2\sin x + C}$

Integration Using Partial Fractions

Using this method of integration involves writing a quotient as a sum of fractions which allows for easier evaluation of the integral in question.

remember The degree of the numerator must be less than the degree of the denominator.

Ex 6) $\int \frac{1}{2x^2 - 5x - 12}dx = \int \frac{1}{(2x+3)(x-4)}dx = \int \left(\frac{-\frac{2}{11}}{2x+3} + \frac{\frac{1}{11}}{x-4}\right)dx$

$\frac{1}{(2x+3)(x-4)} = \frac{A}{2x+3} + \frac{B}{x-4}$

$1 = A(x-4) + B(2x+3)$

if $x=4$: $1 = A(0) + B(11)$

$B = 1/11$

if $x = -\frac{3}{2}$: $1 = A\left(\frac{-11}{2}\right) + B(0)$

$A = -\frac{2}{11}$

$-\frac{2}{11}\left(\frac{1}{2}\right)\ln|2x+3| + \frac{1}{11}\ln|x-4| + C$

$\boxed{-\frac{1}{11}\ln|2x+3| + \frac{1}{11}\ln|x-4| + C}$

Integration of an Improper Integral

Ex 7) $\int_0^\infty \frac{1}{x^2+1}dx$

$\lim_{b\to\infty} \int_0^b \frac{1}{x^2+1}dx = \lim_{b\to\infty}\left(\arctan x + C \Big|_0^b\right)$

$= \lim_{b\to\infty}(\arctan b - \arctan 0)$

$\frac{\pi}{2} - 0 = \frac{\pi}{2}$ $\boxed{\text{converges}}$

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