AP Calculus

Notes **Series**

Integration Review:

- Determine convergence/divergence of a series.
- Write a Taylor polynomial for a function, and use that polynomial for a function and use that polynomial to approximate a value for a function.
- Determine the interval of convergence or radius of convergence.
- Construct a new series from a known series.

Example 1 Determine if the series converges or diverges.

A. $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \dots$ geometric $r = \frac{1}{3}$ $\left|\frac{1}{3}\right| < 1$ converges

B. $\sum_{n=1}^{\infty} \frac{n}{3^n}$ ratio test $\lim_{n\to\infty} \left| \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} \right| = \lim_{n\to\infty} \left| \frac{3^n}{3^{n+1}} \cdot \frac{n+1}{n} \right| = \lim_{n\to\infty} \left| 3^{-1} \cdot \frac{n+1}{n} \right|$

$\frac{1}{3} \cdot 1 = \frac{1}{3} < 1$ converges

C. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ alt. series test

① $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓ ② $\frac{1}{n+1} < \frac{1}{n}$ ✓ converges

D. $\sum_{n=1}^{\infty} \frac{1}{5+2^n}$ compare to $\sum \frac{1}{2^n} = \sum \left(\frac{1}{2}\right)^n$ geometric $r = \frac{1}{2}$ $\left|\frac{1}{2}\right| < 1$ converges

$\lim_{n\to\infty} \frac{\frac{1}{5+2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{5+2^n} = 1$ pos. finite converges by limit comparison

E. $\sum_{n=1}^{\infty} \frac{6}{n^4}$ p-series $p = 4$ $4 > 1$ converges

F. $\sum_{n=2}^{\infty} \frac{1}{n-1}$ compare to $\sum \frac{1}{n}$ p-series $p = 1$ diverges

$\lim_{n\to\infty} \frac{\frac{1}{n-1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n-1} = 1$ pos. finite diverges by limit comparison

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Example 2 Given $f(x) = \sin x$. Write a 3rd degree Taylor polynomial centered around $\frac{\pi}{4}$.

		at $x = \pi/4$
$f(x)$	$\sin x$	$\sqrt{2}/2$
$f'(x)$	$\cos x$	$\sqrt{2}/2$
$f''(x)$	$-\sin x$	$-\sqrt{2}/2$
$f'''(x)$	$-\cos x$	$-\sqrt{2}/2$

$$P_3(x) = \frac{\sqrt{2}}{2} + \frac{\frac{\sqrt{2}}{2}}{1!}\left(x - \pi/4\right)^1 + \frac{\frac{-\sqrt{2}}{2}}{2!}\left(x - \pi/4\right)^2 + \frac{\frac{-\sqrt{2}}{2}}{3!}\left(x - \pi/4\right)^3$$

Interval of Convergence

ratio test

The interval of convergence of a power series: $\sum_{n=0}^{\infty} c_n \cdot (x-a)^n$ is the interval of x-values that can be plugged into the power series to give a convergent series.

The center of the interval of convergence is always the anchor point of the power series, a.

Radius of Convergence

The radius of convergence is half of the length of the interval of convergence.

Example 3 Find the interval of convergence and the radius of convergence: $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n}$

$$\lim_{n\to\infty} \left| \frac{\frac{(x+3)^{n+1}}{n+1}}{\frac{(x+3)^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{(x+3)^{n+1}}{(x+3)^n} \cdot \frac{n}{n+1} \right| = \lim_{n\to\infty} \left| (x+3) \cdot \frac{n}{n+1} \right|$$

$$|x+3| \cdot 1 < 1$$

$$-1 < x+3 < 1$$

$$-4 < x < -2$$

Check end pts:

$x = -4$: $\sum_1 \frac{(-1)^n}{n}$ alt. series converges

$x = -2$: $\sum_1 \frac{(1)^n}{n} = \sum_1 \frac{1}{n}$ p-series p=1 div.

I.O.C. $[-4, -2)$

R.O.C. $= 1$

Known Maclaurin Series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$$

Example 4 Using the Maclaurin series for sin(x), find $\lim_{x\to 0} \frac{\sin x - x}{x^3}$

$$\lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \cancel{-x}}{x^3}$$

$$= \lim_{x\to 0} \left(\frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \ldots \right) = \frac{-1}{3!} = \boxed{-\frac{1}{6}}$$

Example 5 The Maclaurin series for a function $f(x)$ is $1+\frac{x}{2!}+\frac{x^2}{3!}+\frac{x^3}{4!}+\ldots+\frac{x^n}{(n+1)!}+\ldots$

A. Find the first three nonzero terms and the general term of the Maclaurin series for $g(x)=xf(x)$.

$$x\,f(x) = \underbrace{x + \frac{x^2}{2!} + \frac{x^3}{3!}}_{\text{1st 3 terms}} + \frac{x^4}{4!} + \ldots + \underbrace{\frac{x^{n+1}}{(n+1)!}}_{\text{general term}} + \ldots$$

B. Give a formula for $g(x)$ that does not involve a series.

$$g(x) = e^x - 1$$

Error

1. actual error--the difference between the actual function value and the approximation
2. error bound--gives the largest possible error for an approximation

❋ *alternating series error bound*--take |*next omitted term*|

❋ *Lagrange error bound* $\left|R_n(x)\right| \le \frac{f^{n+1}(a)}{(n+1)!}\left|x-c\right|^{n+1}$ max deriv

Where f(x) is centered at c and a is some number between x and c.

Example 6 Find an error bound for cos(0.3) using a 2nd degree Maclaurin polynomial.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \qquad \text{error} < \left|\frac{.3^4}{4!}\right| \to .0003375$$

Example 7 Approximate cos(0.1) using a fourth-degree Maclaurin polynomial. Find the associated Lagrange remainder (error bound). $n=4$ $\to c=0$

$$\left|\frac{1}{(4+1)!}(.1-0)^{4+1}\right|$$

$$8.333\times 10^{-8}$$

$$.00000008333$$