Trigonometric Integrals

- use for integrals of the form $\int \sin^m x \cos^n x \, dx$ and $\int \sec^m x \tan^n x \, dx$ where either m or n is a positive integer
- **goal:** try to break them into combinations of trigonometric integrals to which you can apply the Power Rule

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \cot^2 x = \csc^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

Power-reducing Identities

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

Guidelines for Evaluating Integrals Involving Sine and Cosine

1. If the power of the sine is odd and positive, save one sine factor and convert the remaining factors to cosines. Then, expand and integrate.

$$\int \sin^{\overbrace{2k+1}^{\text{Odd}}} x \cos^n x \, dx = \int \overbrace{(\sin^2 x)^k}^{\text{Convert to cosines}} \cos^n x \overbrace{\sin x \, dx}^{\text{Save for } du} = \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

2. If the power of the cosine is odd and positive, save one cosine factor and convert the remaining factors to sines. Then, expand and integrate.

$$\int \sin^m x \cos^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int \sin^m x \overbrace{(\cos^2 x)^k}^{\text{Convert to sines}} \overbrace{\cos x \, dx}^{\text{Save for } du} = \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

3. If the powers of both the sine and cosine are even and nonnegative, make repeated use of the identities

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

to convert the integrand to odd powers of the cosine. Then proceed as in guideline 2.

Guidelines for Evaluating Integrals Involving Secant and Tangent

1. If the power of the secant is even and positive, save a secant-squared factor and convert the remaining factors to tangents. Then expand and integrate.

$$\int \sec^{\overbrace{2k}^{\text{Even}}} x \tan^n x \, dx = \int \overbrace{(\sec^2 x)^{k-1}}^{\text{Convert to tangents}} \tan^n x \overbrace{\sec^2 x \, dx}^{\text{Save for } du} = \int (1 + \tan^2 x)^{k-1} \tan^n x \sec^2 x \, dx$$

2. If the power of the tangent is odd and positive, save a secant-tangent factor and convert the remaining factors to secants. Then expand and integrate.

$$\int \sec^m x \tan^{\overbrace{2k+1}^{\text{Odd}}} x \, dx = \int \sec^{m-1} x \overbrace{(\tan^2 x)^k}^{\text{Convert to secants}} \overbrace{\sec x \tan x \, dx}^{\text{Save for } du} = \int \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x \, dx$$

3. If there are no secant factors and the power of the tangent is even and positive, convert a tangent-squared factor to a secant-squared factor, then expand and repeat if necessary.

$$\int \tan^n x \, dx = \int \tan^{n-2} x \overbrace{(\tan^2 x)}^{\text{Convert to secants}} \, dx = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

4. If the integral is of the form $\int \sec^m x \, dx$, where m is odd and positive, use integration by parts, as illustrated in Example 5 in the preceding section.

5. If none of the first four guidelines applies, try converting to sines and cosines.

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Example 1 Evaluate: $\int \sin^3 x \cos^4 x dx$ power of sine is odd

$$\int \sin x \cdot \underbrace{\sin^2 x} \cdot \cos^4 x \, dx$$

$$\int \sin x \, (1-\cos^2 x)\cos^4 x \, dx$$

$$\int \sin x \cos^4 x \, dx - \int \sin x \cos^6 x \, dx$$

$u = \cos x$

$\frac{du}{dx} = -\sin x$

$-du = \sin x dx$

$-\int u^4 du = -\frac{1}{5}u^5 + C \qquad -\int u^6 du = -\frac{1}{7}u^7 + C$

$$\boxed{-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C}$$

Example 2 Evaluate: $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^3 x}{\sqrt{\sin x}} dx$ power of cos is odd

$$\int \frac{\cos x \cdot \cos^2 x}{\sqrt{\sin x}} dx = \int \frac{\cos x\,(1-\sin^2 x)}{\sqrt{\sin x}} dx$$

$$= \int \frac{\cos x \, dx}{\sqrt{\sin x}} - \int \frac{\cos x \sin^2 x}{\sqrt{\sin x}} dx$$

$u = \sin x$

$du = \cos x \, dx$

$\int \frac{du}{\sqrt{u}} = \int u^{-1/2} du$

$2u^{1/2}$

$\int \cos x \, (\sin x)^{3/2} dx$

$\int u^{3/2} du = \frac{2}{5} u^{5/2}$

$$2\sqrt{\sin x} - \frac{2}{5}(\sin x)^{5/2} + C \Big|_{\pi/6}^{\pi/3} = \boxed{.239}$$

Example 3 Evaluate: $\int \cos^4 x dx$ cos is even

$$\int \cos^2 x \cdot \cos^2 x \, dx$$

$$\int \left(\frac{1+\cos 2x}{2}\right) \cdot \left(\frac{1+\cos 2x}{2}\right) dx$$

cos is even

$$\int \left(\frac{1}{4} + \frac{\cancel{2}^{1}\cos 2x}{\cancel{4}2} + \frac{\cos^2(2x)}{4}\right) dx$$

$$\int \left(\frac{1}{4} + \frac{\cos 2x}{2} + \frac{1+\cos(4x)}{2\cdot 4}\right) dx$$

$$\int \left(\frac{3}{8} + \frac{\cos 2x}{2} + \frac{\cos 4x}{8}\right) dX$$

$$\boxed{\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C}$$

$u = 2x$

$\frac{du}{dx} = 2$

$\frac{1}{2}du = dx$

$\frac{1}{2}\cdot\frac{1}{2}\int \cos u \, du$

$\frac{1}{4}\sin u$

$u = 4x$

$\frac{du}{dx} = 4$

$\frac{1}{4}du = dx$

$\frac{1}{8}\cdot\frac{1}{4}\int \cos u \, du$

$\frac{1}{32}\sin u$

Example 4 Evaluate: $\int \sec^4 3x \tan^3 3x dx$ power of secant is even

$$\int \sec^2 3x \cdot \sec^2 3x \tan^3 3x \, dx$$

$$\int \sec^2 3x \left(\tan^2 3x + 1\right) \tan^3 3x \, dx$$

$$\int \sec^2 3x \tan^5 3x \, dx + \int \sec^2 3x \tan^3 3x \, dx$$

$u = \tan 3x$

$\frac{du}{dx} = \sec^2 3x \cdot 3$

$\frac{1}{3}du = \sec^2 3x dx$

$\frac{1}{3}\int u^5 du = \frac{1}{3}\cdot\frac{1}{6}u^6$

$\frac{1}{3}\int u^3 du$

$\frac{1}{3}\cdot\frac{1}{4}u^4$

$$\boxed{\frac{1}{18}\tan^6 3x + \frac{1}{12}\tan^4 3x + C}$$