Trigonometric Substitution

- objective is to eliminate the radical in the integrand

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$1 + \cot^2 x = \csc^2 x$

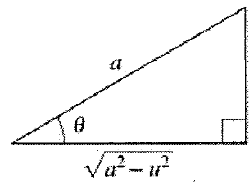
$\tan^2 x + 1 = \sec^2 x$

Trigonometric Substitution ($a > 0$)

1. For integrals involving $\sqrt{a^2 - u^2}$, let

   $u = a \sin\theta$. $\sin\theta = \frac{u}{a}$

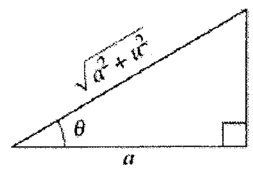
   Then $\sqrt{a^2 - u^2} = a\cos\theta$, where $-\pi/2 \le \theta \le \pi/2$.



2. For integrals involving $\sqrt{a^2 + u^2}$, let

   $u = a \tan\theta$. $\tan\theta = \frac{u}{a}$

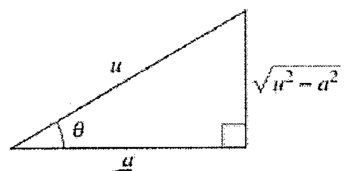
   Then $\sqrt{a^2 + u^2} = a\sec\theta$, where $-\pi/2 < \theta < \pi/2$.



3. For integrals involving $\sqrt{u^2 - a^2}$, let

   $u = a \sec\theta$. $\sec\theta = \frac{u}{a}$

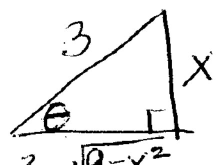
   Then $\sqrt{u^2 - a^2} = \pm a\tan\theta$, where $0 \le \theta < \pi/2$ or $\pi/2 < \theta \le \pi$. Use the positive value if $u > a$ and the negative value if $u < -a$.



$\tan\theta = \frac{\sqrt{u^2 - a^2}}{a}$

NOTE The restrictions on θ ensure that the function that defines the substitution is one-to-one. In fact, these are the same intervals over which the arcsine, arctangent, and arcsecant are defined.

Example 1 Evaluate: $\int \frac{dx}{x^2\sqrt{9-x^2}}$ type 1



$9 = a^2 \quad x^2 = u^2$

$a = 3 \quad x = u$

$u = a\sin\theta$

$x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$ $\quad \cot\theta = \frac{\sqrt{9-x^2}}{x}$

$\sqrt{9-x^2} = 3\cos\theta$

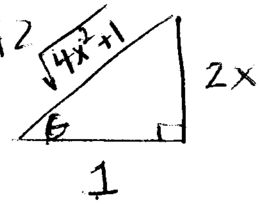
$\int \frac{3\cos\theta\, d\theta}{(3\sin\theta)^2(3\cos\theta)}$

$\int \frac{d\theta}{9\sin^2\theta}$

$\int \frac{1}{9}\csc^2\theta\, d\theta$

8 $\quad -\frac{1}{9}\cot\theta + C = -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C$

Example 2 Evaluate: $\int \frac{dx}{\sqrt{4x^2+1}}$ type #2



$$\int \frac{\frac{1}{2}\sec^2\theta \, d\theta}{\sec\theta}$$

$4x^2 = u^2 \quad 1 = a^2$

$2x = u \quad 1 = a$

$\sec\theta = \frac{\sqrt{4x^2+1}}{1}$

$u = a\tan\theta$

$$\frac{1}{2}\int \sec\theta \, d\theta$$

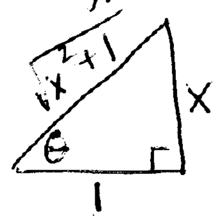
$2x = \tan\theta$

$x = \frac{1}{2}\tan\theta$

$$\frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C \qquad dx = \frac{1}{2}\sec^2\theta \, d\theta$$

$$\boxed{\frac{1}{2}\ln\left|\sqrt{4x^2+1} + 2x\right| + C}$$

Example 3 Evaluate: $\int \frac{dx}{\left(x^2+1\right)^{\frac{3}{2}}} = \int \frac{dx}{\left(\sqrt{x^2+1}\right)^3}$ type #2



$\sin\theta = \frac{x}{\sqrt{x^2+1}}$

$$\int \frac{\sec^2\theta \, d\theta}{\left(\sec\theta\right)^3}$$

$x^2 = u^2 \quad 1 = a^2$

$x = u \quad 1 = a$

$u = a\tan\theta$

$\sec\theta = \frac{\sqrt{x^2+1}}{1}$

$$\int \frac{1}{\sec\theta} \, d\theta$$

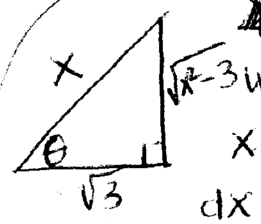
$x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$\int \cos\theta \, d\theta$$

$$\sin\theta + C = \boxed{\frac{x}{\sqrt{x^2+1}} + C}$$

Example 4 Evaluate: $\int_{\sqrt{3}}^{2} \frac{\sqrt{x^2-3}}{x} dx$ type # 3



$u = a\sec\theta$

$u^2 = x^2 \quad a^2 = 3$

$x = \sqrt{3}\sec\theta$

$u = x \quad a = \sqrt{3}$

$dx = \sqrt{3}\sec\theta\tan\theta \, d\theta$

$$\int \frac{\sqrt{3}\tan\theta \cdot \sqrt{3}\sec\theta\tan\theta \, d\theta}{\sqrt{3}\sec\theta}$$

$a\tan\theta = \sqrt{u^2 - a^2}$

$\sqrt{3}\tan\theta = \sqrt{x^2-3}$

$$\sqrt{3}\int \tan^2\theta \, d\theta$$

when $x = 2$: $2 = \sqrt{3}\sec\theta$ when $x = \sqrt{3}$: $\sqrt{3} = \sqrt{3}\sec\theta$

$\frac{2}{\sqrt{3}} = \sec\theta \qquad 1 = \sec\theta$

$$\sqrt{3}\int \left(\sec^2\theta - 1\right) d\theta$$

$\cos\theta = 1$

$\cos\theta = \frac{\sqrt{3}}{2} \qquad \theta = \cos^{-1}(1)$

$$\sqrt{3}\left[\tan\theta - \theta\right] + C \Big|^{\pi/6} = \sqrt{3}\left[\frac{1}{\sqrt{3}} - \frac{\pi}{6} - (0-0)\right]$$

$\theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \quad \theta = \pi/6 \qquad 0$