Optimization notes

Example 1 Find two positive numbers whose product is 192, and the sum of the first plus three times the second is a minimum.

Let $x = 1^{st}\#$ $\quad y = 2^{nd}\#$

$xy = 192 \rightarrow y = \frac{192}{x}$

$S = x + 3y$ minimize

$S = x + 3\left(\frac{192}{x}\right) = x + 576x^{-1}$

$S' = 1 - 576x^{-2} = 1 - \frac{576}{x^2} = \frac{x^2 - 576}{x^2}$

undef when $x = 0$

$\frac{x^2 - 576}{x^2} = 0$

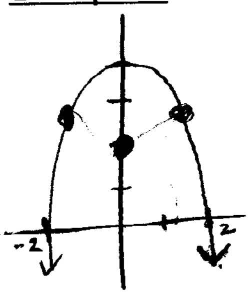
$x^2 - 576 = 0$

$x = \pm\sqrt{576} = \pm 24$

$+ \quad - \quad - \quad +$; $\cancel{-24}$, $\cancel{0}$, $24 \leftarrow$ min

$y = \frac{192}{24} = 8$ $\quad \boxed{24 \,\&\, 8}$

Example 2 Which points on the graph of $y = 4 - x^2$ are closest to the point (0,2)?



minimize the distance between $(0,2)$ and (x, y)

$d = \sqrt{(x-0)^2 + (y-2)^2}$

$d = \sqrt{x^2 + (4 - x^2 - 2)^2}$ $\quad (2 - x^2)$

$d = \sqrt{x^2 + 4 - 4x^2 + x^4}$

$d = \sqrt{-3x^2 + x^4 + 4}$ minimize

deriv: $-6x + 4x^3 = 0$

$2x(-3 + 2x^2) = 0$

$2x = 0 \qquad -3 + 2x^2 = 0$

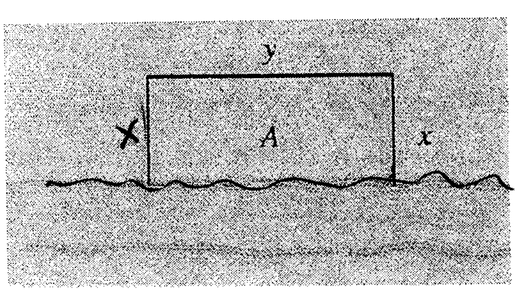
$x = 0 \qquad x^2 = \frac{3}{2}$

$x = \pm\sqrt{\frac{3}{2}}$

$- \quad + \quad - \quad +$; $-\sqrt{3/2}$, 0, $\sqrt{3/2}$

$\left(-\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$ $\quad \left(\sqrt{\frac{3}{2}}, \frac{5}{2}\right)$

Example 3 A farmer wants to fence in a rectangular region. A river runs along one of the sides of the region, so the farmer only needs to put the fence on three of the sides. If the farmer has 200 yards of fencing to use, find the largest possible area which can be enclosed by the fence. What are the dimensions of the region?



$A = xy$ maximize

$2x + y = 200$

$y = 200 - 2x$

$A = x(200 - 2x) = 200x - 2x^2$

$A' = 200 - 4x = 0$

$x = 50$

$+ \quad -$; $50 \leftarrow$ max

$y = 200 - 2(50)$
$= 100$

dim: 50yd × 100yd

area = 5000 yd²

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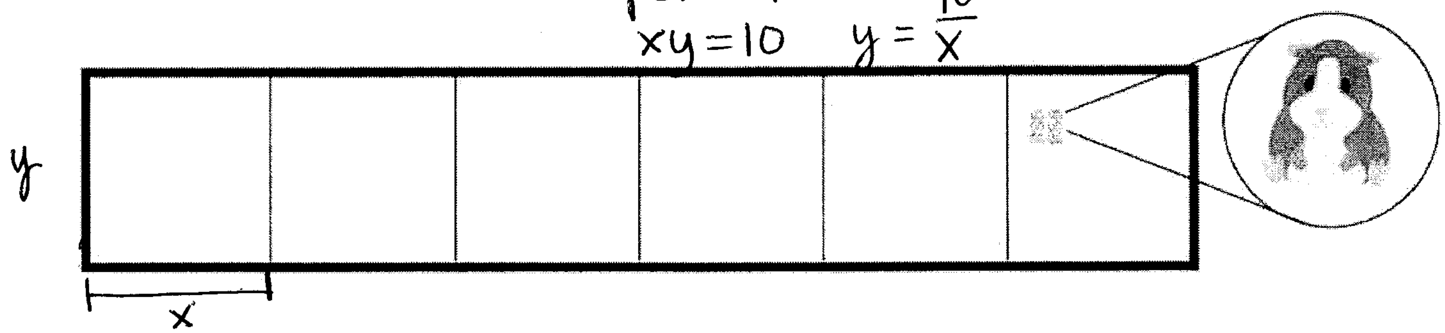
Example 4 Your dream of becoming a hamster breeder has finally come true. You are constructing a set of rectangular pens in which to breed your furry friends. The overall area you are working with is 60 square feet, and you want to divide the area up into six pens of equal size as shown below. each pen $= 10\ ft^2$

$xy = 10 \quad y = \frac{10}{x}$



The cost of the outside fencing is \$10 a foot. The inside fencing costs \$5 a foot. You wish to minimize the cost of the fencing. Find the exact dimensions of each pen that will minimize the cost of the breeding ground. What is the total cost?

$$C = 10(12x + 2y) + 5(5y)$$
$$C = 120x + 20y + 25y$$
$$C = 120x + 45y = 120x + 45\left(\frac{10}{x}\right) = 120x + 450x^{-1}$$

minimize

$$C' = 120 - 450x^{-2} = 120 - \frac{450}{x^2} = \frac{120x^2 - 450}{x^2}$$

undef. when $x = 0$

$$\frac{120x^2 - 450}{x^2} = 0$$
$$120x^2 - 450 = 0$$
$$x^2 = \frac{15}{4}$$
$$x = \pm\sqrt{\frac{15}{4}} = \pm\frac{\sqrt{15}}{2}$$

$$\begin{array}{ccccccc} & + & & - & & + & \\ \hline \cancel{-\frac{\sqrt{15}}{2}} & & \cancel{0} & & \frac{\sqrt{15}}{2} & & \end{array}$$

$\leftarrow$ min

$$y = \frac{10}{\frac{\sqrt{15}}{2}} = \frac{20}{\sqrt{15}}$$

dim: $\frac{\sqrt{15}}{2}$ ft $\times$ $\frac{20}{\sqrt{15}}$ ft

$$\text{cost} = 120\left(\frac{\sqrt{15}}{2}\right) + \frac{450}{\frac{\sqrt{15}}{2}}$$
$$= \$464.76$$