AP Calculus AB

Notes: The Conjugate Method/Trig Limits

Recall from Precalc that the conjugate of $a + bi$ is $\underline{a - bi}$ The conjugate of $x - \sqrt{2}$ is $\underline{x + \sqrt{2}}$.

$\frac{2-2}{0} = \frac{0}{0}$

1) How can the conjugate help us? When trying to evaluate $\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x}$, we multiply by the conjugate of $\sqrt{x+4} - 2$ which is $\underline{\sqrt{x+4} + 2}$. Watch what happens:

$$\lim_{x \to 0} \left(\frac{\sqrt{x+4} - 2}{x}\right)\left(\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\right) = \lim_{x \to 0} \frac{x + 4 + 2\sqrt{x+4} - 2\sqrt{x+4} - 4}{x(\sqrt{x+4} + 2)} = \lim_{x \to 0} \frac{1}{\sqrt{x+4} + 2}$$

$$\frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

More examples:

2) $\lim_{t \to 0} \left(\frac{\sqrt{2-t} - \sqrt{2}}{t}\right)\left(\frac{\sqrt{2-t} + \sqrt{2}}{\sqrt{2-t} + \sqrt{2}}\right) =$

3) $\cancel{\lim_{x \to 0} \frac{\sqrt{2x+3} - \sqrt{3}}{x}} =$

4) $\lim_{h \to 0} \left(\frac{h}{\sqrt{3h+7} - \sqrt{7}}\right)\left(\frac{\sqrt{3h+7} + \sqrt{7}}{\sqrt{3h+7} + \sqrt{7}}\right)$

$$\lim_{t \to 0} \frac{2 - t - 2}{t(\sqrt{2-t} + \sqrt{2})} = \lim_{t \to 0} \frac{-1}{\sqrt{2-t} + \sqrt{2}} = \frac{-1}{\sqrt{2} + \sqrt{2}} = \frac{-1}{2\sqrt{2}}$$

$$\lim_{h \to 0} \frac{h(\sqrt{3h+7} + \sqrt{7})}{3h + 7 - 7}$$

$$\lim_{h \to 0} \frac{\sqrt{3h+7} + \sqrt{7}}{3} = \boxed{\frac{2\sqrt{7}}{3}}$$

Evaluating limits involving Trig:

1) $\lim_{x \to \pi} \frac{\sin x + \cos x}{2 \cos x} = \underline{\frac{1}{2}}$

$$\frac{0 + -1}{2(-1)} = \frac{-1}{-2}$$

2) $\lim_{\theta \to 0} \sin\theta \cos\theta = \underline{0}$

$0 \cdot 1$

3) $\lim_{\theta \to \frac{5\pi}{6}} \sin\theta \cos\theta = \underline{-\frac{\sqrt{3}}{4}}$

$\frac{1}{2} \cdot -\frac{\sqrt{3}}{2}$

4) $\lim_{\theta \to \frac{\pi}{2}} \underbrace{\tan\theta \cos^2\theta \sec\theta} = \underline{1}$

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\sin\theta}{\cos\theta} \cdot \cos^2\theta \cdot \frac{1}{\cos\theta}$$

$$\lim_{\theta \to \pi/2} \sin\theta$$

5) $\lim_{x \to \pi} \frac{1}{4 \cos x} = \underline{-\frac{1}{4}}$

$\frac{1}{4(-1)}$

6) $\lim_{\theta \to 0} \tan\theta = \underline{0}$

$\frac{0}{1}$

7) $\lim_{\theta \to \frac{3\pi}{4}} \cot\theta = \underline{-1}$

$$\frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}$$

8) $\lim_{\theta \to \frac{-\pi}{2}} \sin\theta \cot\theta = \underline{0}$

$-1 \cdot \frac{0}{-1}$

4

4 Special trig limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$

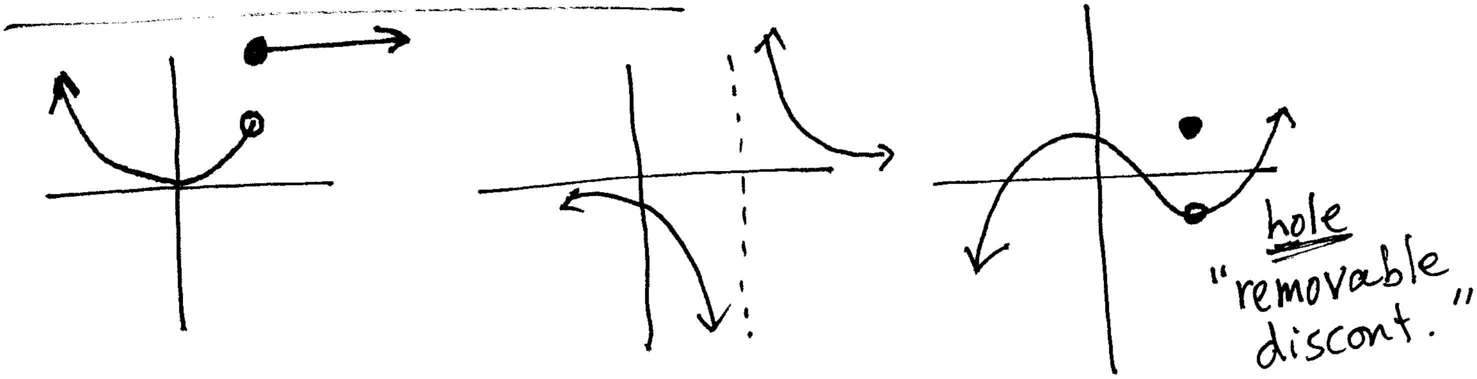
A $\displaystyle \lim_{x\to 0} \frac{\sin 4x}{x} = \lim_{x\to 0} 4 \cdot \frac{\sin 4x}{4x} = 4 \cdot 1 = \boxed{4}$

B $\displaystyle \lim_{x\to 0} \frac{\sin 10x}{\sin 3x} = \lim_{x\to 0} \frac{\sin 10x}{10x} \cdot \frac{3x}{\sin 3x} \cdot \frac{10x}{3x}$

$\displaystyle = 1 \cdot 1 \cdot \frac{10}{3} = \boxed{\frac{10}{3}}$

C $\displaystyle \lim_{x\to 0} \frac{3(1-\cos x)}{x} = 3 \cdot 0 = \boxed{0}$

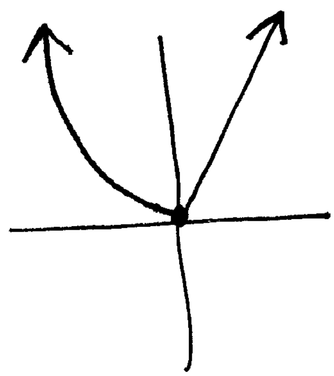
Discontinuities



A function is continuous at a point $x=c$ if:

1. $f(c)$ exists (there's a y-value)
2. $\lim_{x \to c} f(x)$ exists (left & right-sided limits agree)
3. $f(c) = \lim_{x \to c} f(x)$ (part 1 = part 2)

EX1 $f(x) = \begin{cases} x^2, & x \le 0 \\ 2x, & x > 0 \end{cases}$ Is $f(x)$ cont. at $x=0$? Justify.



yes

1. $f(0) = 0$
2. $\lim_{x \to 0} f(x) = 0$

   $\lim_{x \to 0^-} f(x) = 0$

   $\lim_{x \to 0^+} f(x) = 0$
3. $f(0) = \lim_{x \to 0} f(x)$ ✓

EX2 $f(x) = \begin{cases} x+1, & x<2 \\ k(x-5)^2, & x \geq 2 \end{cases}$ where k is a constant

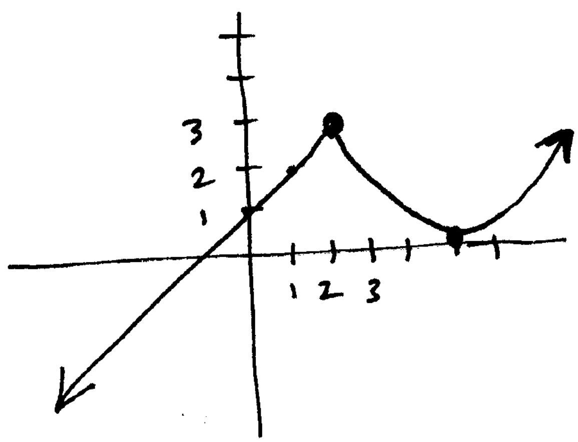
Find k so that f(x) is continuous. Sketch f(x).

when $x = 2$ $\quad x+1 = k(x-5)^2$

$3 = k(-3)^2$

$3 = 9k$

$k = \frac{3}{9} = \frac{1}{3}$



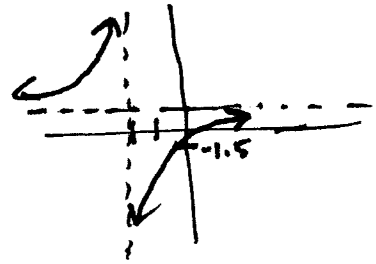
vertical asymptotes to find: look for zeros of denom.

to justify: find one-sided limits around discont.

EX3 $f(x) = \frac{x^2-9}{x^2+5x+6} = \frac{(x+3)(x-3)}{(x+2)(x+3)} = \frac{x-3}{x+2}$

v.a. at $x=-2$ (hole at $x=-3$)

$$\lim_{x \to -2^-} f(x) = \infty$$



$$\lim_{x \to -2^+} f(x) = -\infty$$

$$\lim_{x \to -2} f(x) = \text{DNE}$$