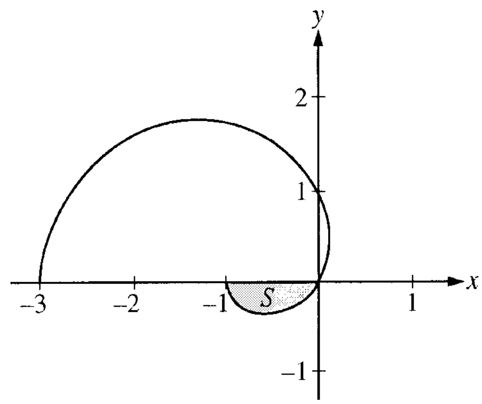
AP® CALCULUS BC
2009 SCORING GUIDELINES (Form B)

Question 4



The graph of the polar curve $r = 1 - 2\cos\theta$ for $0 \le \theta \le \pi$ is shown above. Let S be the shaded region in the third quadrant bounded by the curve and the x-axis.

(a) Write an integral expression for the area of S.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(c) Write an equation in terms of x and y for the line tangent to the graph of the polar curve at the point where $\theta = \frac{\pi}{2}$. Show the computations that lead to your answer.

(a) $r(0) = -1$; $r(\theta) = 0$ when $\theta = \frac{\pi}{3}$.

Area of $S = \frac{1}{2}\int_0^{\pi/3} (1 - 2\cos\theta)^2\, d\theta$

$1 - 2\cos\theta = 0$
$\cos\theta = \frac{1}{2}$
$\theta = \pi/3$

$2 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{cases}$

$y =$
$= (1-2\cos\theta)(\cos\theta) + \sin\theta(r2\sin\theta)$

(b) $x = r\cos\theta$ and $y = r\sin\theta$

$x' = r\sin\theta + \cos\theta \cdot \frac{dr}{d\theta}$

$\frac{dr}{d\theta} = 2\sin\theta$ $\quad x = (1-2\cos\theta)\cos\theta$
$x' = (1-2\cos\theta)(\sin\theta) + \cos\theta(2\sin\theta)$

$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta = 4\sin\theta\cos\theta - \sin\theta$

$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta = 2\sin^2\theta + (1 - 2\cos\theta)\cos\theta$

$x = (1-2\cos\theta)\cdot\cos\theta$

$\frac{dx}{d\theta} = (1-2\cos\theta)\cdot -\sin\theta + \cos\theta(2\sin\theta) = -\sin\theta + 4\sin\theta\cos\theta$

$4 : \begin{cases} 1 : \text{uses } x = r\cos\theta \text{ and } y = r\sin\theta \\ 1 : \frac{dr}{d\theta} \\ 2 : \text{answer} \end{cases}$

$x = r\cos\theta = 0 \quad y = r\sin\theta = 1$ ($\theta = \pi/2$)

(c) When $\theta = \frac{\pi}{2}$, we have $x = 0$, $y = 1$.

$x = (1-2\cos\theta)\cdot\cos\theta \quad y = (1-2\cos\theta)\sin\theta$

Slope $= \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \left.\frac{dy/d\theta}{dx/d\theta}\right|_{\theta=\frac{\pi}{2}} = -2$

The tangent line is given by $y = 1 - 2x$.

$3 : \begin{cases} 1 : \text{values for } x \text{ and } y \\ 1 : \text{expression for } \frac{dy}{dx} \\ 1 : \text{tangent line equation} \end{cases}$

part b: $\dfrac{dy/d\theta}{dx/d\theta}$ at $\pi/2$: $\dfrac{2(1)^2 + (1-0)\cdot 0 = 2}{4(1)(0) - 1}$ $\quad 2$

#2

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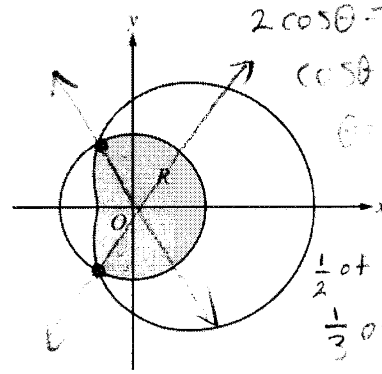
2007 SCORING GUIDELINES

Question 3



$3+2\cos\theta = 2$
$2\cos\theta = -1$
$\cos\theta = -1/2$
$\theta = \cos^{-1}(-1/2)$
$2\pi/3,\ 4\pi/3$

$\frac{1}{2}$ of circle +
$\frac{1}{3}$ of $\frac{1}{2}$ of circle

The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\, d\theta$
$= 10.370$

$\frac{1}{2} + \frac{1}{6}$

$\frac{3+1}{6} = \frac{4}{6} = \frac{2}{3}$ of circle w/ $r = 2$

4 :
- 1 : area of circular sector
- 2 : integral for section of limaçon
  - 1 : integrand
  - 1 : limits and constant
- 1 : answer

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

$r = 3+2\cos\theta \quad \frac{dr}{d\theta} = -2\sin\theta$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

at $\pi/3 \approx -1.732$

2 :
- 1 : $\left.\frac{dr}{dt}\right|_{\theta=\pi/3}$
- 1 : interpretation

(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

$\frac{dy}{d\theta} = (3+2\cos\theta)(\cos\theta) + (\sin\theta)\cdot -2\sin\theta$
$= 3\cos\theta + 2\cos^2\theta - 2\sin^2\theta$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

at $\pi/3$ = [illegible]

3 :
- 1 : expression for y in terms of θ
- 1 : $\left.\frac{dy}{dt}\right|_{\theta=\pi/3}$
- 1: interpretation

#3

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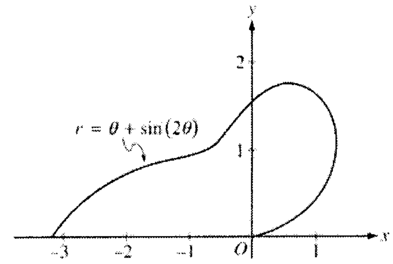
Question 2



The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) $\text{Area} = \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = \boxed{4.382}$

$3: \begin{cases} 1: \text{limits and constant} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

$x = r\cos\theta$

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$

$\boxed{\theta = 2.786}$

$2: \begin{cases} 1: \text{equation} \\ 1: \text{answer} \end{cases}$

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

$2: \begin{cases} 1: \text{information about } r \\ 1: \text{information about the curve} \end{cases}$

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\boxed{\theta = \frac{\pi}{3}}$.

$2: \begin{cases} 1: \theta = \frac{\pi}{3} \text{ or } 1.047 \\ 1: \text{answer with justification} \end{cases}$

#4

$y^2 = 1-(x-1)^2$
$y = \pm\sqrt{1-(x-1)^2}$

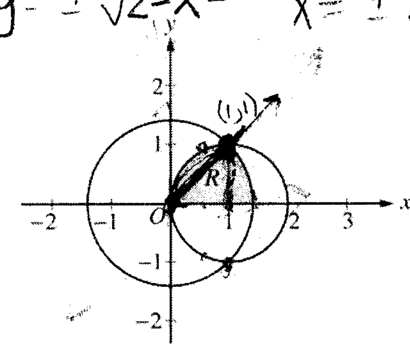
$(x-1)^2 = 1-y^2$
$x-1 = \pm\sqrt{1-y^2}$
$x = 1 \pm \sqrt{1-y^2}$

$y^2 = 2-x^2$
$y = \pm\sqrt{2-x^2}$

$x^2 = 2-y^2$
$x = \pm\sqrt{2-y^2}$

Question 2

The figure above shows the graphs of the circles $x^2 + y^2 = 2$ and $(x-1)^2 + y^2 = 1$. The graphs intersect at the points $(1,1)$ and $(1,-1)$. Let R be the shaded region in the first quadrant bounded by the two circles and the x-axis.



(a) Set up an expression involving one or more integrals with respect to x that represents the area of R.

(b) Set up an expression involving one or more integrals with respect to y that represents the area of R.

(c) The polar equations of the circles are $r = \sqrt{2}$ and $r = 2\cos\theta$, respectively. Set up an expression involving one or more integrals with respect to the polar angle θ that represents the area of R.

(a) Area $= \displaystyle\int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

OR

Area $= \dfrac{1}{4}(\pi \cdot 1^2) + \displaystyle\int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

3 :
- 1 : integrand for larger circle
- 1 : integrand or geometric area for smaller circle
- 1 : limits on integral(s)

Note: $< -1 >$ if no addition of terms

(b) Area $= \displaystyle\int_0^1 \left(\sqrt{2-y^2} - \left(1-\sqrt{1-y^2}\right)\right)dy$

3 :
- 1 : limits
- 2 : integrand
  - $< -1 >$ reversal
  - $< -1 >$ algebra error in solving for x
  - $< -1 >$ add rather than subtract
  - $< -2 >$ other errors

(c) Area $= \displaystyle\int_0^{\pi/4} \frac{1}{2}\left(\sqrt{2}\right)^2 d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

OR

Area $= \dfrac{1}{8}\pi\left(\sqrt{2}\right)^2 + \displaystyle\int_{\pi/4}^{\pi/2} \frac{1}{2}(2\cos\theta)^2\,d\theta$

3 :
- 1 : integrand or geometric area for larger circle
- 1 : integrand for smaller circle
- 1 : limits on integral(s)

Note: $< -1 >$ if no addition of terms

#1 a. $1 - 2\cos\theta = 0$

$\cos\theta = \frac{1}{2}$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = \pi/3$

$$\frac{1}{2}\int_0^{\pi/3} (1-2\cos\theta)^2 \, d\theta$$ 1pt.

b. $x = r\cos\theta \qquad y = r\sin\theta$

$x = (1-2\cos\theta)\cos\theta \quad y = (1-2\cos\theta)\sin\theta$ 1pt.

$\frac{dx}{d\theta} = (1-2\cos\theta)(-\sin\theta) + \cos\theta(2\sin\theta)$ 1pt.

2pt. $\frac{dy}{d\theta} = (1-2\cos\theta)\cos\theta + \sin\theta(2\sin\theta)$

c. slope $= \frac{dy}{dx} = \left.\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right|_{\theta=\frac{\pi}{2}} = \frac{(1-2\cdot 0)\cdot 0 + 1(2\cdot 1)}{(1-2\cdot 0)\cdot -1 + 0(2\cdot 1)} = \frac{2}{-1} = -2$

1pt.

point (x, y)

$x = 1 \cdot \cos\frac{\pi}{2} = 0$

$y = 1 \cdot \sin \pi/2 = 1$

$(0, 1)$ 1pt.

$\boxed{y - 1 = -2(x - 0)}$ 1pt.

#2 a. 90° $\frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2)^2 = \pi$ $* 2 \rightarrow 2\pi$

30° $\frac{1}{3}(\pi) = \frac{\pi}{3}$ $* 2 \rightarrow \frac{2\pi}{3}$

$\frac{1}{2}\int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} (3+2\cos\theta)^2 \, d\theta \longrightarrow 1.993$

$\boxed{10.370}$

1pt.

$2\left[\frac{1}{2}\int_0^{2\pi/3} (2)^2 \, d\theta\right]$

b. $\frac{dr}{dt} = \frac{dr}{d\theta} = -2\sin\theta\Big|_{\theta=\pi/3} = -2\left(\sin\frac{\pi}{3}\right) = -2\cdot\frac{\sqrt{3}}{2} = -\sqrt{3}$

$\boxed{-1.732}$

1pt.

$\frac{dr}{dt}$ is neg $\Rightarrow$ radius is decreasing

$\Rightarrow$ particle is getting closer to the origin

1pt.

c. $\frac{dy}{dt} = \frac{dy}{d\theta} = (3+2\cos\theta)(\cos\theta) + \sin\theta(-2\sin\theta)\Big|_{\theta=\pi/3}$

$y = r\sin\theta = (3+2\cos\theta)\sin\theta$ 1pt. $= .5$ 1pt.

$\frac{dy}{dt}$ is positive $\Rightarrow$ particle is moving up & away from x-axis

1pt.

Polar FRQ

#3. a. $\frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = 4.382$

b. $x = r\cos\theta$

$-2 = (\theta + \sin(2\theta))\cos\theta$

$\theta = 2.786$

c. $\frac{dr}{d\theta}$ negative means the radius is decreasing on $(\frac{\pi}{3}, \frac{2\pi}{3})$.

if radius is decreasing, the curve is getting closer to the origin

d. $r = \theta + \sin(2\theta)$ greatest distance = abs. max

$\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$

$1 + 2\cos(2\theta) = 0$ $0 \le \theta \le \frac{\pi}{2}$

$\cos(2\theta) = -\frac{1}{2}$ $0 \le 2\theta \le \pi$

$2\theta = \cos^{-1}(-\frac{1}{2})$

$2\theta = \frac{2\pi}{3}$

$\theta = \frac{\pi}{3}$

θ	r
0	0
$\pi/3$	$\pi/3 + \sin\frac{2\pi}{3} = 1.913$
$\pi/2$	$\pi/2 + \sin\pi = 1.571$

← largest $\boxed{\theta = \frac{\pi}{3}}$

#4 a. $\int_0^1 \sqrt{1-(x-1)^2}\,dx + \int_1^{\sqrt{2}} \sqrt{2-x^2}\,dx$

$(x-1)^2 = 1-y^2$ $\quad y^2 = 1-(x-1)^2$ $\quad y^2 = 2-x^2$ $\quad x^2 = 2-y^2$

$x-1 = \pm\sqrt{1-y^2}$ $\quad y = \pm\sqrt{1-(x-1)^2}$ $\quad y = \pm\sqrt{2-x^2}$ $\quad x = \sqrt{2-y^2}$

$x = 1 \pm \sqrt{1-y^2}$

b. $\int_0^1 \left(\sqrt{2-y^2} - \left(1+\sqrt{1-y^2}\right)\right) dy$

c. $2\cos\theta = \sqrt{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}, \frac{7\pi}{4}$

$\underbrace{\frac{1}{2}\int_0^{\pi/4} (\sqrt{2})^2\,d\theta}_{} + \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\,d\theta$

OR

$\frac{1}{8}\pi(\sqrt{2})^2 = \frac{\pi}{4}$