last unit -- series of constants
this unit -- series of functions

Power Series centered at $x=c$

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \dots a_n(x-c)^n + \dots$$

if $c=0$: $a_0 + a_1 x + a_2 x^2 + \dots a_n x^n + \dots$

Convergence of a Power series Centered at $x=c$

think domain

one of the following occurs:

1. the series converges only at c — R.O.C. $= 0$
2. the series converges absolutely for all x — R.O.C. $= \infty$
3. the series converges for an interval of values — R.O.C. $= \frac{1}{2}$(length of interval)

radius of convergence ("R") = the distance from the midpoint of the interval of convergence to the endpoint

→ use the ratio test

EX1 Find the I.O.C. and the R.O.C.

Ⓐ $\sum_{n=0}^{\infty} n\left(\frac{2x-5}{3}\right)^n = 0 + \left(\frac{2x-5}{3}\right)^1 + 2\left(\frac{2x-5}{3}\right)^2 + \dots$

$$\lim_{n\to\infty}\left|\frac{(n+1)\left(\frac{2x-5}{3}\right)^{n+1}}{n\left(\frac{2x-5}{3}\right)^n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n}\cdot\left(\frac{2x-5}{3}\right)^1\right|$$

$$= 1\cdot\left|\frac{2x-5}{3}\right| < 1$$ converges if

$$-1 < \frac{2x-5}{3} < 1$$

$$-3 < 2x-5 < 3$$

$$2 < 2x < 8$$

$$1 < x < 4$$

Check endpts

$x=1$: $\sum n\left(\frac{2(1)-5}{3}\right)^n = \sum (-1)^n n$

alt. series

$\lim_{n\to\infty} n = \infty$ div. (n^{th} term test)

$x=4$: $\sum n\left(\frac{2(4)-5}{3}\right)^n = \sum n$

$\lim_{n\to\infty} n = \infty$ div. n^{th} term test

I.O.C. $(1,4)$

R.O.C. $= \frac{1}{2}(3) = \frac{3}{2}$

(B) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \ldots$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)^2}}{\frac{x^n}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n^2}{(n+1)^2} \right| = \lim_{n\to\infty} \left| x^1 \cdot \frac{n^2}{(n+1)^2} \right|$$

$$= |x| \cdot 1$$

converges if $|x| < 1$

$-1 < x < 1$

Check endpts:

$x=-1$: $\displaystyle\sum \frac{(-1)^n}{n^2}$ alt. series $\displaystyle\lim_{n\to\infty} \frac{1}{n^2} = 0$ conv.

$x=1$: $\displaystyle\sum \frac{(1)^n}{n^2} = \sum \frac{1}{n^2}$ p-series $p = 2 > 1$ conv.

I.O.C. $[-1, 1]$

R.O.C. $= \frac{1}{2}(2) = 1$

(C) $\displaystyle\sum_{n=0}^{\infty} (2n)! \left(\frac{x}{2}\right)^n = 1 + 2\left(\frac{x}{2}\right) + 24\left(\frac{x}{2}\right)^2 + \ldots$

$$\frac{(2n+2)(2n+1)(2n)(2n-1)\ldots(1)}{(2n)(2n-1)(2n-2)\ldots(1)}$$

$$\lim_{n\to\infty} \left| \frac{(2n+2)!\left(\frac{x}{2}\right)^{n+1}}{(2n)!\left(\frac{x}{2}\right)^n} \right| = \lim_{n\to\infty} \left| (2n+2)(2n+1)\left(\frac{x}{2}\right)^1 \right| = \infty$$

diverges

I.O.C. only at the center $x=0$ R.O.C. $= 0$

D $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{6} + \frac{x^5}{120} + \dots$

$2n+3-2n-1$

$2n+3-(2n+1)$

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{x^{2n+1}} \cdot \frac{(2n+1)!}{(2n+3)!} \right|$$

$$\frac{(2n+1)(2n)(2n-1)\dots(1)}{(2n+3)(2n+2)\cancel{(2n+1)}\cancel{(2n)}\dots(1)}$$

$$= \lim_{n\to\infty} \left| x^2 \cdot \frac{1}{(2n+3)(2n+2)} \right|$$

$$= |x^2| \cdot 0 = 0 < 1$$

I.O.C. converges for all x

$(-\infty, \infty)$

R.O.C $= \infty$

EX2 Given $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, find $f'(x)$.

long way $\left\{ \begin{array}{l} f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots = e^x \\ f'(x) = 0 + 1 + x + \frac{1}{2}x^2 + \dots = e^x \end{array} \right.$

shortcut

↳ use power rule! $\sum_{n=1}^{\infty} \frac{n \cdot x^{n-1}}{n!}$

EX3 Given $\sum \frac{(-1)^{n+1}(x-5)^n}{n \cdot 5^n}$. Find $f'(x)$, $f''(x)$, & $\int f(x)dx$

$$f'(x) = \sum \frac{(-1)^{n+1} \cdot n(x-5)^{n-1}}{n \cdot 5^n} = \sum \frac{(-1)^{n+1}(x-5)^{n-1}}{5^n}$$

$$f''(x) = \sum \frac{(-1)^{n+1}(n-1)(x-5)^{n-2}}{5^n}$$

$$\int f(x)dx = \sum \frac{(-1)^{n+1}}{n \cdot 5^n} \frac{(x-5)^{n+1}}{(n+1)} + C$$

---

* calculus on series
  I.O.C. will be the same as original series <u>except</u> must recheck the endpoints