Ratio Test

- good for factorials & $\sqrt{\ }$

$\sum a_n$. Find $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$

1. If $L<1$, series converges absolutely.
2. If $L>1$ (or DNE), series diverges.
3. If $L=1$, the ratio test doesn't apply.

EX1 Determine convergence.

A) $\sum \frac{1}{n!}$

$$\lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{\cancel{n(n-1)(n-2)\cdots(1)}}{(n+1)\cancel{(n)(n-1)\cdots(1)}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\right| = 0$$

$0<1$ series converges by the Ratio Test

B) $\sum \frac{n^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{(n+1)^1\cdot n^n}\right|$$

$\frac{n!}{(n+1)!} \to \frac{1}{n+1}$

$$= \lim_{n\to\infty}\left|\frac{(n+1)^n}{n^n}\right| = \lim_{n\to\infty}\left|\left(\frac{n+1}{n}\right)^n\right|$$

1^∞ use L'Hopital's

$= e \quad e>1$

series diverges by the Ratio Test

ⓒ $\sum \frac{(-1)^n \sqrt{n}}{n+1}$

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1} \sqrt{n+1}}{n+2}}{\frac{(-1)^n \sqrt{n}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{(n+1)}{(n+2)} \right| = 1$$

$\to (-1)^1$

$1 \cdot 1 \cdot 1$

ratio test fails

alt. series test

① $\lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0$ ✓

② $\frac{\sqrt{n+1}}{n+2} < \frac{\sqrt{n}}{n+1}$ ✓

series converges by the A.S.T.

The Root Test — good for n^{th} powers

$\sum a_n$. Find $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$

① If $L < 1$, series converges absolutely.

② If $L > 1$, series diverges.

③ ~~If~~ $L = 1$, root test does not apply.

D $\sum \left(\frac{n}{2n+1}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{n}{2n+1}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{n}{2n+1}\right| = \frac{1}{2} \quad \frac{1}{2}<1$$

series converges by the Root Test

E $\sum \frac{e^{2n}}{n^n} = \sum \left(\frac{e^2}{n}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{e^2}{n}\right)^n\right|} = \lim_{n\to\infty}\left|\frac{e^2}{n}\right| = 0 \quad 0<1$$

series converges by the Root Test