AP Calculus BC Review Notes Derivative Rules and Techniques

You should be able to . . .

- **Apply the limit definition of a derivative:**

  If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$.

  OR

  Suppose f is a function defined on an open interval containing a number p. The derivative of f at p is denoted by $f'(p)$ and is defined by

  $f'(p) = \lim\limits_{x\to p}\dfrac{f(x)-f(p)}{x-p}$ provided this limit exists.

- **Find the derivative of a function using rules (basic, product, quotient, chain, etc.):**

  If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

  1. $(cf)' = cf'(x)$
  2. $(f \pm g)' = f'(x) \pm g'(x)$
  3. $(fg)' = f'g + fg'$ – **Product Rule**
  4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$ – **Quotient Rule**
  5. $\dfrac{d}{dx}(c) = 0$
  6. $\dfrac{d}{dx}(x^n) = nx^{n-1}$ – **Power Rule**
  7. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$
     This is the **Chain Rule**

Common Derivatives

$\dfrac{d}{dx}(x) = 1$

$\dfrac{d}{dx}(\sin x) = \cos x$

$\dfrac{d}{dx}(\cos x) = -\sin x$

$\dfrac{d}{dx}(\tan x) = \sec^2 x$

$\dfrac{d}{dx}(\sec x) = \sec x \tan x$

$\dfrac{d}{dx}(\csc x) = -\csc x \cot x$

$\dfrac{d}{dx}(\cot x) = -\csc^2 x$

$\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{d}{dx}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

$\dfrac{d}{dx}(a^x) = a^x \ln(a)$

$\dfrac{d}{dx}(e^x) = e^x$

$\dfrac{d}{dx}(\ln(x)) = \dfrac{1}{x},\ x > 0$

$\dfrac{d}{dx}(\ln|x|) = \dfrac{1}{x},\ x \neq 0$

$\dfrac{d}{dx}(\log_a(x)) = \dfrac{1}{x \ln a},\ x > 0$

- **Find higher-order derivatives, derivatives implicitly and derivatives using logarithmic differentiation.**

  Logarithmic Differentiation
  It is often advantageous to use logarithms to differentiate certain functions.
  1. Take ln of both sides
  2. Differentiate
  3. Solve for y'
  4. Substitute for y
  5. Simplify

- **Find a derivative of an inverse.** $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

- **Find the equation of a tangent or normal line.**

reminder of point-slope form: $y - y_1 = m(x - x_1)$

Slope

Ordered pair

Ex 1) Find:

a) $\lim_{\Delta x \to 0} \frac{\sin\left(\frac{\pi}{3}+\Delta x\right)-\sin\left(\frac{\pi}{3}\right)}{\Delta x}$

$f(x) = \sin x$
$f'(x) = \cos x$
$f'\left(\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$

b) $\lim_{h \to 0} \frac{\csc\left(\frac{\pi}{4}+h\right)-\csc\left(\frac{\pi}{4}\right)}{h}$

$f(x) = \csc x$
$f'(x) = -\csc x \cot x$
$f'\left(\frac{\pi}{4}\right) = -\csc\frac{\pi}{4}\cot\frac{\pi}{4}$
$-\frac{2}{\sqrt{2}} \cdot 1 = -\frac{2}{\sqrt{2}}$
$-\frac{2\sqrt{2}}{2} = \boxed{-\sqrt{2}}$

c) $\lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}+h\right) - \cos\left(\frac{\pi}{2}\right)}{h}$

0

$f(x) = \cos x$
$f'(x) = -\sin x$
$f'\left(\frac{\pi}{2}\right) = \boxed{-1}$

d) $\lim_{h \to 0} \frac{\sqrt{9+h}-3}{h}$

$\sqrt{9}$

$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2}x^{-1/2}$ $f'(9) = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

e) $\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \sin\frac{\pi}{4}}{x - \frac{\pi}{4}}$

$f(x) = \sin x$
$f'(x) = \cos x$
$f'\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$

f) $\lim_{x \to 9} \frac{\sqrt{x}-3}{x-9}$

$f(x) = \sqrt{x}$
$f'(x) = \frac{1}{2}x^{-1/2}$
$f'(9) = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

Ex 2) Find the derivative:

a) $y = 3x\cos(4x+2)$

$y' = (3x)(-\sin(4x+2))(4) + (\cos(4x+2))(3)$
$= -12x\sin(4x+2) + 3\cos(4x+2)$

b) $y = \frac{x^2-3x+4}{\sin(3x)}$

$y' = \frac{\sin(3x) \cdot (2x-3) - (x^2-3x+4)(3\cos\ldots}{(\sin 3x)^2}$

c) $f(x) = \sin^{-1}(2x)$

$\frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$

d) $g(x) = 5^x$

$g'(x) = 5^x \cdot \ln 5$

Ex 3) Find $f^{(4)}(x)$ if $f(x) = 5x^8 - 7x^6 + 5x^4 - 2x^3 + 9$

$f'(x) = 40x^7 - 42x^5 + 20x^3 - 6x^2$
$f''(x) = 280x^6 - 210x^4 + 60x^2 - 12x$
$f'''(x) = 1680x^5 - 840x^3 + 120x - \ldots$
$f^{(4)}(x) = 8400x^4 - 2520x^2 + 120\ldots$

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Ex 4) Differentiable functions f and g have the values shown in the table.

x	f	f'	g	g'
0	2	1	5	−4
1	3	2	3	−3
2	5	3	1	−2
3	10	4	0	−1

a. If $h(x)=\sqrt{f(x)}$, find $h'(3)$. $(f(x))^{1/2}$

A) $\frac{1}{4}$ B) $\frac{1}{2\sqrt{10}}$ C) 2 D) $\frac{2}{\sqrt{10}}$ E) $4\sqrt{10}$

$$h'(x)=\frac{1}{2}(f(x))^{-1/2}(f'(x))$$

$$h'(3)=\frac{f'(3)}{2\sqrt{f(3)}}=\frac{4}{2\sqrt{10}}=\frac{2}{\sqrt{10}}$$

b. If $h(x)=f(g(x))$, find $h'(1)$.

A) −12 B) −6 C) 4 D) 6 E) 12

$$h'(x)=f'(g(x))\cdot g'(x)$$

$$h'(1)=f'(\underbrace{g(1)}_{3})\cdot g'(1)=4\cdot -3=-12$$

Ex 5) Find $\frac{dy}{dx}$ if $y=\left(\frac{x^2+1}{x^2-1}\right)^{\frac{1}{3}}$

$$\ln y=\ln\left(\frac{x^2+1}{x^2-1}\right)^{1/3}$$

$$\ln y=\frac{1}{3}\cdot\left[\ln(x^2+1)-\ln(x^2-1)\right]$$

deriv: $\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left[\frac{1}{x^2+1}\cdot 2x-\frac{1}{x^2-1}\cdot 2x\right]$

$$\frac{dy}{dx}=y\cdot\frac{1}{3}\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$

$$\frac{dy}{dx}=\left(\frac{x^2+1}{x^2-1}\right)^{1/3}\cdot\frac{1}{3}\left[\frac{2x}{x^2+1}-\frac{2x}{x^2-1}\right]$$

Ex 6) Find the derivative: $xy^2+5\ln x=14-e^y$

$$x\cdot 2y\frac{dy}{dx}+y^2\cdot 1+5\cdot\frac{1}{x}=-e^y\cdot\frac{dy}{dx}$$

$$y^2+\frac{5}{x}=-2xy\frac{dy}{dx}-e^y\frac{dy}{dx}$$

$$y^2+\frac{5}{x}=\frac{dy}{dx}(-2xy-e^y)$$

$$\frac{dy}{dx}=\frac{y^2+\frac{5}{x}}{-2xy-e^y}$$

Ex 7) A function f and its derivative take on the values shown in the table. If g is the inverse of f, find $g'(6)$.

x	$f(x)$	$f'(x)$
2	6	1/3
6	8	3/2

$$g'(6)=\frac{1}{f'(g(6))}=\frac{1}{f'(2)}=\frac{1}{\frac{1}{3}}=3$$

$(6, 2)$ 6 x-coord for g

$(2, 6)$ 6 y-coord for f

Ex 8) Find the equation of the tangent line and the normal line through the point (2, 59) for the curve $f(x)=8x^3+2x-9$.

$$f'(x)=24x^2+2$$

$$f'(2)=96+2=98$$

tangent: $y-59=98(x-2)$

normal: $y-59=-\frac{1}{98}(x-2)$