Remainder Theorem
If f(x) is divided by x − k, then the remainder is f(k).

Factor Theorem
If f(k) = 0, then x − k is a factor of f(x).

Example 4 $p(x) = 2x^2 + 3x + 4$ Divide by x − 3.

$$\begin{array}{r|rrr} 3 & 2 & 3 & 4 \\ & \downarrow & 6 & 27 \\ \hline & 2 & 9 & 31 \end{array} \qquad \frac{2x^2+3x+4}{x-3} = 2x+9+\frac{31}{x-3}$$

Example 5 $p(x) = 2x^3 - 3x^2 - 5x - 12$ Is x − 3 a factor?

yes

$$\begin{array}{r|rrrr} 3 & 2 & -3 & -5 & -12 \\ & \downarrow & 6 & 9 & 12 \\ \hline & 2 & 3 & 4 & 0 \end{array}$$ remainder

Example 6 $p(x) = 3x^3 + 4x^2 - 5x - 2$ Is x + 2 a factor? Is x + 1 a factor?

yes no

$$\begin{array}{r|rrrr} -2 & 3 & 4 & -5 & -2 \\ & \downarrow & -6 & 4 & 2 \\ \hline & 3 & -2 & -1 & 0 \end{array}$$ rem.

$$\begin{array}{r|rrrr} -1 & 3 & 4 & -5 & -2 \\ & \downarrow & -3 & -1 & 6 \\ \hline & 3 & 1 & -6 & 4 \end{array}$$ rem.

Example 7 $p(x) = x^3 - 3x - 2$ Is x − 2 a factor?

yes

$$\begin{array}{r|rrrr} 2 & 1 & 0 & -3 & -2 \\ & \downarrow & 2 & 4 & 2 \\ \hline & 1 & 2 & 1 & 0 \end{array}$$ rem

ICM Notes on the Rational Root Theorem

Rational Root Theorem

Given the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$.

Let p = factors of a_0 and let q = factors of a_n.

Then $\frac{p}{q}$ are ***possible*** rational roots.

constant   leading coefficient

Examples Find the roots.

1. $f(x) = 2x^3 + 11x^2 - 7x - 6$

$p = 1, 2, 3, 6$

$q = 1, 2$

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

possible rational roots

$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & \downarrow & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & 0 \end{array}$$

$2x^2 + 13x + 6 = 0$

$(2x + 1)(x + 6) = 0$

$2x + 1 = 0 \quad x + 6 = 0$

$x = -\frac{1}{2} \quad x = -6$

$\boxed{\text{roots: } 1, -\tfrac{1}{2}, -6}$

2. $f(x) = 3x^3 + 4x^2 - 5x - 2$

$p: 1, 2$

$q: 1, 3$

$\frac{p}{q}: \pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$

$$\begin{array}{r|rrrr} 2 & 3 & 4 & -5 & -2 \\ & \downarrow & 6 & 20 & 30 \\ \hline & 3 & 10 & 15 & 28 \end{array}$$

$$\begin{array}{r|rrrr} 1 & 3 & 4 & -5 & -2 \\ & \downarrow & 3 & 7 & 2 \\ \hline & 3 & 7 & 2 & 0 \end{array}$$

$3x^2 + 7x + 2 = 0$

$(3x + 1)(x + 2) = 0$

$3x + 1 = 0 \quad x + 2 = 0$

$x = -\frac{1}{3} \quad x = -2$

$\boxed{\text{roots: } 1, -\tfrac{1}{3}, -2}$

3. $f(x) = x^5 - 5x^4 + 12x^3 - 24x^2 + 32x - 16$

$p: 1, 2, 4, 8, 16$

$q: 1$

$\frac{p}{q} = \pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$$\begin{array}{r|rrrrrr} 1 & 1 & -5 & 12 & -24 & 32 & -16 \\ & \downarrow & 1 & -4 & 8 & -16 & 16 \\ \hline & 1 & -4 & 8 & -16 & 16 & 0 \end{array}$$

$$\begin{array}{r|rrrrr} 2 & 1 & -4 & 8 & -16 & 16 \\ & \downarrow & 2 & -4 & 8 & -16 \\ \hline & 1 & -2 & 4 & -8 & 0 \end{array}$$

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4. $f(x) = x^3 - 8x^2 + 9x + 6$

p: 1, 2, 3, 6
q: 1

$\frac{p}{q} = \pm 1, \pm 2, \pm 3, \pm 6$

$$\begin{array}{r|rrrr} 2 & 1 & -8 & 9 & 6 \\ & \downarrow & 2 & -12 & -6 \\ \hline & 1 & -6 & -3 & 0 \end{array}$$

$x^2 - 6x - 3 = 0$

$a = 1 \quad b = -6 \quad c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-3)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{48}}{2} = \frac{6 \pm 4\sqrt{3}}{2} = 3 \pm 2\sqrt{3}$$

$\boxed{\text{roots: } 2,\ 3+2\sqrt{3},\ 3-2\sqrt{3}}$

5. $f(x) = 2x^3 - 8x^2 + 15x - 27$

p: 1, 3, 9, 27
q: 1, 2

$\frac{p}{q} = \pm 1, \pm 3, \pm 9, \pm 27, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{27}{2}$

$\boxed{\text{real root: } 3}$

$$\begin{array}{r|rrrr} 3 & 2 & -8 & 15 & -27 \\ & \downarrow & 6 & -6 & 27 \\ \hline & 2 & -2 & 9 & 0 \end{array}$$

$2x^2 - 2x + 9 = 0$

$a = 2 \quad b = -2 \quad c = 9$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(9)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{-68}}{4} = \frac{2 \pm 2i\sqrt{17}}{4}$$

$$= \frac{1}{2} \pm \frac{1}{2} i\sqrt{17}$$

complex

$\underbrace{x^3 - 2x^2} + 4x - 8 = 0$

$x^2(x-2) + 4(x-2) = 0$

$(x-2)(x^2+4) = 0$

$x - 2 = 0 \qquad x^2 + 4 = 0$

$x = 2 \qquad x^2 = -4$

$x = \pm\sqrt{-4} = \pm 2i$

imaginary

real roots: 2, 1

multiplicity of 2