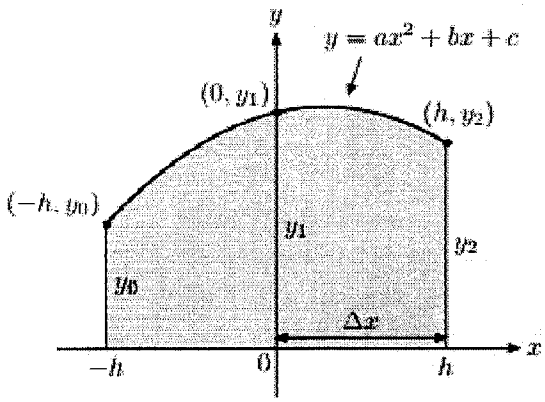
Numerical Integration--Simpson's Rule

<u>**Simpson's Rule**</u>--a numerical method that approximates the value of a definite integral by using quadratic polynomials.

Thomas Simpson
(1710 - 1761)

<u>Deriving the formula</u>

Given a parabola with equation $y = ax^2 + bx + c$ passing through the three points: $(-h, y_0)$, $(0, y_1)$, (h, y_2).



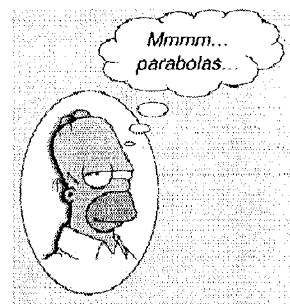
$$A = \int_{-h}^{h} (ax^2 + bx + c)\, dx =$$

$$= \tfrac{1}{3}ax^3 + \tfrac{1}{2}bx^2 + cx + C \Big|_{-h}^{h}$$

$$= \tfrac{1}{3}ah^3 + \tfrac{1}{2}bh^2 + ch + C - \left(-\tfrac{1}{3}ah^3 + \tfrac{1}{2}bh^2 - ch + C\right)$$

$$= \tfrac{2}{3}ah^3 + 2ch = \frac{h}{3}\left(2ah^2 + 6c\right)$$

Since the points $(-h, y_0)$, $(0, y_1)$, and (h, y_2) are on the parabola, they satisfy $y = ax^2 + bx + c$. Therefore,



$y_0 = ah^2 - bh + c$

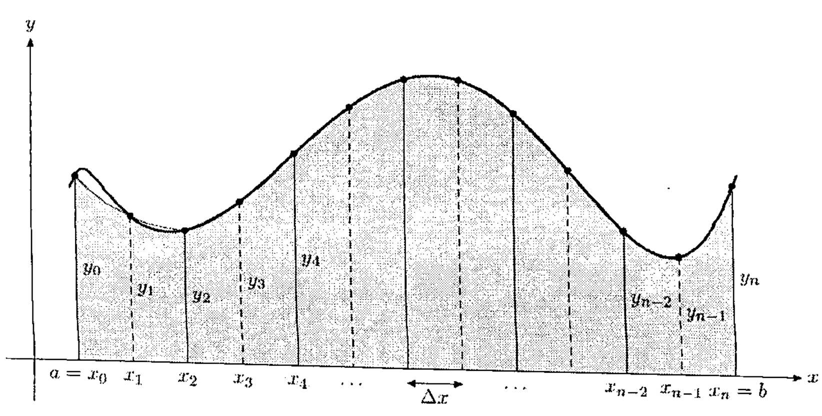
$y_1 = c$

$y_2 = ah^2 + bh + c$

Observe that $y_0 + 4y_1 + y_2 = ah^2 \cancel{- bh} + c + 4c + ah^2 \cancel{+ bh} + c$

$$= 2ah^2 + 6c$$

Therefore, the area under the parabola is $A = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right)$



We can estimate the integral by adding the areas under the parabolic arcs through three successive points.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0+4y_1+y_2\right)+\frac{\Delta x}{3}\left(y_2+4y_3+y_4\right)+\cdots+\frac{\Delta x}{3}\left(y_{n-2}+4y_{n-1}+y_n\right)$$

By simplifying, we obtain Simpson's rule formula.

$$\boxed{\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left(y_0+4y_1+2y_2+4y_3+2y_4+\cdots+4y_{n-1}+y_n\right)}$$

* n is even

$$\Delta x = \frac{b-a}{n} \qquad \frac{\Delta x}{3} \to \frac{b-a}{3n}$$

Example 1 Approximate $\int_0^{\pi} \sin x\,dx$ if n = 4 and if n = 8. Then find the exact value.

$0 \quad \pi/8 \quad \pi/4 \quad 3\pi/8 \quad \pi/2 \quad 5\pi/8 \quad 3\pi/4 \quad 7\pi/8 \quad \pi$

$$n=4: \quad \frac{\pi-0}{3(4)}\left(\sin 0+4\sin\tfrac{\pi}{4}+2\sin\tfrac{\pi}{2}+4\sin\tfrac{3\pi}{4}+\sin\pi\right)\approx 2.005$$

$$n=8: \quad \frac{\pi-0}{3(8)}\left(\sin 0+4\sin\tfrac{\pi}{8}+2\sin\tfrac{\pi}{4}+4\sin\tfrac{3\pi}{8}+2\sin\tfrac{\pi}{2}+4\sin\tfrac{5\pi}{8}+2\sin\tfrac{3\pi}{4}+4\sin\tfrac{7\pi}{8}+\sin\pi\right)\approx 2.0003$$

$$\int_0^{\pi}\sin x\,dx = -\cos x + C\Big|_0^{\pi} = -\cos\pi + C - \left(-\cos 0 + C\right) = 1+1 = 2$$

Example 2 Approximate $\int_1^3 x^3\,dx$ if n = 6. Then find the exact value.

$1 \quad 4/3 \quad 5/3 \quad 2 \quad \frac{7}{3} \quad \frac{8}{3} \quad 3$

$$n=6: \quad \frac{3-1}{3(6)}\left((1)^3+4\left(\tfrac{4}{3}\right)^3+2\left(\tfrac{5}{3}\right)^3+4(2)^3+2\left(\tfrac{7}{3}\right)^3+4\left(\tfrac{8}{3}\right)^3+(3)^3\right)=20$$

$$\int_1^3 x^3\,dx = \frac{1}{4}x^4 + C\Big|_1^3 = \frac{81}{4}-\frac{1}{4} = 20$$

Example 3 Approximate the integral $\int_0^{16} f(x)\,dx$ using Simpson's Rule:

x	y
0	2
4	3
8	1
12	8
16	6

4 subint n=4

2

$$\frac{16-0}{3(4)}\left(2+4\cdot 3+2\cdot 1+4\cdot 8+6\right)$$

72