test review unit 3

1. $\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{(-1)^{n+1}x^{2(n+1)+1}}{(2(n+1))!}}{\dfrac{(-1)^n x^{2n+1}}{(2n)!}}\right| = \lim\limits_{n\to\infty}\left|(-1)^1 \cdot x^{2n+2} \cdot \dfrac{1}{(2n+2)(2n+1)}\right| = 0$

I.O.C. $(-\infty, \infty)$ R.O.C. $= \infty$

2. a) $\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{x^{n+2}}{(n+2)^2}}{\dfrac{x^{n+1}}{(n+1)^2}}\right| = \lim\limits_{n\to\infty}\left|x^1 \cdot \dfrac{(n+1)^2}{(n+2)^2}\right| = |x| < 1$

$-1 < x < 1$

check endpts:

$x = -1$: $\sum \dfrac{(-1)^{n+1}}{(n+1)^2}$ alt. series conv

$x = 1$: $\sum \dfrac{(1)^{n+1}}{(n+1)^2} = \sum \dfrac{1}{(n+1)^2}$ limit comp to p-series $\sum \frac{1}{n^2}$ $p = 2$ conv.

$[-1, 1]$

b) $f'(x) = \sum \dfrac{(n+1)x^n}{(n+1)^2} = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n+1}$

1^{st} 4 terms: $1 + \dfrac{x}{2} + \dfrac{x^2}{3} + \dfrac{x^3}{4}$

general: $\dfrac{x^n}{n+1}$

$\lim\limits_{n\to\infty}\left|\dfrac{\dfrac{x^{n+1}}{n+2}}{\dfrac{x^n}{n+1}}\right| = \lim\limits_{n\to\infty}\left|x^1 \cdot \dfrac{n+1}{n+2}\right|$ $|x| < 1$

$-1 < x < 1$

not needed see part A

check endpts:

$x = -1$: $\sum \dfrac{(-1)^n}{n+1}$ alt. series conv

$x = 1$: $\sum \dfrac{(1)^n}{n+1}$ limit comp p-series $\sum \frac{1}{n}$ div

I.O.C. $[-1, 1)$

3a. know $\frac{1}{1-x} = \sum x^n$

$$\frac{1}{1-(-x^2)} = \sum (-x^2)^n = \sum (-1)^n x^{2n}$$

1st 4 terms: $1 - x^2 + x^4 - x^6$

general: $(-1)^n x^{2n}$

b. $g(x) = \arctan x$

$f(x) = \frac{1}{1+x^2}$ is $g'(x)$

$$g(x) = \int \frac{1}{1+x^2}\,dx = \sum (-1)^n \frac{x^{2n+1}}{2n+1}$$

1st 4 terms: $x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$

general: $(-1)^n \frac{x^{2n+1}}{2n+1}$

c. $\arctan(\frac{1}{3}) \approx \frac{1}{3} - \frac{(\frac{1}{3})^3}{3}$

$\approx \boxed{.321}$

term# ($x=\frac{1}{3}$)	value
1	1/3
2	.0123 ← .001
3	.000823
4	.0000653

4. know $\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \approx \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\sin(x^3) \approx \sum_{n=0}^{\infty} \boxed{\frac{(-1)^n (x^3)^{2n+1}}{(2n+1)!}} \approx \boxed{x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!}}$$

general term: ↑

5. $g(x) = \frac{x}{1+2x}$

know $\frac{1}{1-x} = \sum x^n$

build $\frac{1}{1-(-2x)} = \sum (-2x)^n = \sum (-1)^n (2x)^n$ $2^n \cdot x^n$

build $x \cdot \frac{1}{1+2x} = \sum (-1)^n (2)^n x^n \cdot x^1 = \sum_{n=0}^{\infty} \boxed{(-1)^n 2^n x^{n+1}}$

$\boxed{x - 2x^2 + 4x^3 - 8x^4}$ gen. term

6. $\frac{f''(0)}{2!} = 0$

$\frac{f'''(0)}{3!} = -5$

$\frac{f^{(5)}(0)}{5!} = 4$

$\boxed{\begin{array}{l} f''(0) = 0 \\ f'''(0) = -30 \\ f^{(5)}(0) = 480 \end{array}}$

7. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}$

$\lim_{x \to 0} \frac{\left(\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots\right)\right)}{x}$

$\lim_{x \to 0} \frac{2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)}{x} = \lim_{x \to 0} 2\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \dots\right)$

$= \boxed{2}$

check L'Hopitals: $\lim_{x \to 0} \frac{e^x - e^{-x}}{x} = e^0 + e^0 = 2$

$\lim_{x \to 0} \frac{e^x + e^{-x}}{1}$

8. $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\int_0^1 \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!}}{x}\,dx$

$= \int_0^1 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!}\right)dx = x - \frac{x^3}{3!\cdot 3} + \frac{x^5}{5!\cdot 5} + C\Big|_0^1$

$= 1 - \frac{1}{18} + \frac{1}{600} \approx .946$

9. a) $\boxed{P_3(x) = -5 + 2(x-3) - \frac{7(x-3)^2}{2!} + \frac{9(x-3)^3}{3!}}$

$f(2.6) \approx P_3(2.6) = \boxed{-6.456}$

b) error bound $= \frac{|5(2.6-3)^4|}{4!} = .005333 = \frac{2}{375}$

$P_3(2.6) = -6.456 + .005333 = -6.451$

$= -6.456 - .005333 = -6.461$

so $f(2.6)$ is between

so it can't equal -6

c) $g(x) = f(x^2+3) = -5 + 2(x^2+3-3) - 7(x^2+3-3)^2 / 2!$

$= \boxed{-5 + 2x^2 - \frac{7}{2}x^4}$

d) $g'(x) = 4x - 14x^3$

$2x(2 - 7x^2) = 0$

$x = 0 \quad x^2 = \frac{2}{7} \quad x = \pm\sqrt{\frac{2}{7}}$

[illegible]

10. a) $f(4) = 2$ given

$f'(4) = \frac{-1}{3\cdot 2} = -\frac{1}{6}$

$f''(4) = \frac{2!}{3^2(3)} = \frac{2}{27}$

$f'''(4) = \frac{-1\cdot 3!}{3^3(4)} = \frac{-6}{3^3(4)} = -\frac{1}{18}$

$P_3(x) = 2 - \frac{1}{6}(x-4) + \frac{\frac{2}{27}(x-4)^2}{2!} - \frac{\frac{1}{18}(x-4)^3}{3!}$

$= 2 - \frac{1}{6}(x-4) + \frac{1}{27}(x-4)^2 - \frac{1}{108}(x-4)^3$

b) $1 + \sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{3^n (n+1)}$

$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)}}{\frac{(-1)^n(x-4)^n}{3^n(n+1)}} \right| = \lim_{n\to\infty} \left| (-1)^1 (x-4)^1 \frac{n+1}{n+2} \cdot 3^{-1} \right|$

$\frac{1}{3}|x-4| < 1$ $\quad -1 < \frac{1}{3}(x-4) < 1$

c) $f(5) \approx P_3(5)$ use 1st 3 terms

term	value	
1	2	
2	$-1/6$	
3	$1/27$	$.037$
4	$-1/108$	$.009$

$2 - 1/6 + 1/27 = 1.870$

error $\le |-1/108| \quad 1/108 \quad .009$

11. $\frac{dy}{dx} = x + 2xy = x(1+2y)$

$u = 1+2y$
$du = 2\,dy$
$\frac{1}{2}du = dy$

$\frac{dy}{1+2y} = x\,dx$

$\frac{1}{2}\ln|1+2y| = \frac{1}{2}x^2 + C$

$\frac{1}{2}\ln|1+2(1)| = \frac{1}{2}(0)^2 + C$

$\frac{1}{2}\ln 3 = C$

$\frac{1}{2}\ln|1+2y| = \frac{1}{2}x^2 + \frac{1}{2}\ln 3$

$\ln|1+2y| = x^2 + \ln 3$

$1+2y = 3e^{x^2}$

$2y = 3e^{x^2} - 1$

$y = \frac{3e^{x^2} - 1}{2}$

12. $\frac{1-\cos(2t)}{2t} \cdot 2 = \boxed{\frac{1-\cos 2t}{t}}$

13. $\ln y + e^x = y$

$\frac{1}{y}\frac{dy}{dx} + e^x = 1\frac{dy}{dx}$

$e^x = \frac{dy}{dx}\left(1-\frac{1}{y}\right)$

$\frac{dy}{dx} = \frac{e^x}{1-\frac{1}{y}} = \frac{y(e^x)}{y-1}$

14. $y = e^x = \sum \frac{x^n}{n!}$

$e^{1/4 x} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4}x)^n}{n!} = 1 + \frac{1}{4}x + \frac{\frac{1}{16}x^2}{2!} + \frac{\frac{1}{64}x^3}{3!} + \frac{\frac{1}{256}x^4}{4!}$

$= 1 + \frac{1}{4}x + \frac{1}{32}x^2 + \frac{1}{384}x^3 + \dots$

15. $\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{b\to\infty} \int_1^b \frac{\ln x}{x}\,dx$ $\quad u = \ln x$ $\quad \frac{du}{dx} = \frac{1}{x}$ $\quad \int u\,du$

$\lim_{b\to\infty} \left(\frac{1}{2}(\ln x)^2 + c \Big|_1^b \right) = \lim_{b\to\infty} \frac{1}{2}(\ln b)^2 - \frac{1}{2}(\ln 1)^2$

$\infty - 0$

diverges.

16. A. $\lim_{n\to\infty} n^2 = \infty$ div. n^{th} term test

B. $\lim_{n\to\infty} \frac{2n^2-1}{n^2+1} = 2$ div. n^{th} term test

C. $\lim_{n\to\infty} \left| \frac{\frac{k(n+1)^2}{(n+1)!}}{\frac{kn^2}{n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \frac{n!}{(n+1)!} \right| = 0 < 1$

conv.

D $\lim_{n \to \infty} \dfrac{\frac{1}{n^2-3}}{\frac{1}{n^2}} = \lim_{n \to \infty} \dfrac{n^2}{n^2-3} = 1$ pos. finite

conv. limit comparison to $\sum \frac{1}{n^2}$

p-series p = 2

conv.

E alt. series $\lim_{n \to \infty} \dfrac{1}{\ln n} = 0$ conv.