AP Calculus BC **REVIEW INFINITE SERIES**

For Problems 1-5, be sure to justify your conclusion.

1. Does the sequence $a_n = \left\{\frac{3n}{2n-1}\right\}$ converge or diverge?

$\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2}$ limit exists $\Rightarrow$ sequence converges

2. Does the series $\sum_{n=1}^{\infty} \frac{3n}{2n-1}$ converge or diverge?

n^{th} term test $\lim_{n\to\infty} \frac{3n}{2n-1} = \frac{3}{2} \neq 0$ diverges

3. Use the n-th term test on the series $\sum_{n=1}^{\infty} \frac{6n}{4n^2+1}$ and state its conclusion.

$\lim_{n\to\infty} \frac{6n}{4n^2+1} = 0$ n^{th} term test does not apply

4. Can you find the sum of the series $\sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}}$? If so, find that sum.

$\sum_{n=1}^{\infty} \frac{2^n}{5^n 5^1} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n \cdot \frac{1}{5} = \frac{2}{25} + \frac{4}{125} + \frac{8}{625} + \dots$

$r = \frac{2}{5}$ $a_1 = \frac{2}{25}$ sum $= \frac{\frac{2}{25}}{1-\frac{2}{5}} = \frac{2}{25} \cdot \frac{5}{3} = \frac{2}{15}$

5. Given the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$, find its sum.

telescoping

$\left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{4}-\frac{1}{6}\right) + \dots = 1 + \frac{1}{2} = \frac{3}{2}$

6. Match the series with its behavior by circling the appropriate word.

a. $\sum_{n=1}^{\infty} \frac{1}{n^e}$ p-series $p = e > 1$ converge diverge

b. $\sum_{n=1}^{\infty} \frac{1}{n^{\ln 2}}$ p-series $p = \ln 2 < 1$ converge diverge

7. Find the value of r (if it exists) in the series $3 - \frac{9}{2} + \frac{27}{4} - \dots$

a. $-\frac{3}{2}$ b. $-\frac{2}{3}$ c. $\frac{2}{3}$ d. $\frac{3}{2}$ e. no r exists

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8. How many terms of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ are needed to approximate the sum of the series with an error that is less than 0.001?

a. 3 **b.** 4 **c.** 5 **d.** 6 **e.** 7

term #	1	2	3	4	5	6
value	1	$\frac{1}{16}$	$\frac{1}{81}$	$\frac{1}{256}$	$\frac{1}{625}$	$\frac{1}{1296}$
		.0625	.012	.004	.0016	.00077

9. If the series $\sum_{n=1}^{\infty}\frac{1}{n^{1.01}}$ converges, which of the following series also converges?

a. $\sum_{n=1}^{\infty}\frac{1}{n}$ p=1 div.

b. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ p=½ div.

c. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^{1.01}}$ alt. series conv.

d. $\sum_{n=1}^{\infty}\frac{1}{n^{0.09}}$ p=.09 div

e. $\sum_{n=1}^{\infty}\frac{1}{n^{0.01}}$ p=.01 div

10. Find the sum of the series $\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots$

a. $\frac{2}{3}$ **b.** $\frac{4}{5}$ **c.** $\frac{6}{7}$ **d.** 1 **e.** does not exist

11. Which of the following series is absolutely convergent?

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^{0.99}}$ $\sum\left|\frac{(-1)^{n+1}}{n^{.99}}\right| = \sum\frac{1}{n^{.99}}$ p-series div; $\sum\frac{(-1)^{n+1}}{n^{.99}}$ alt. series conv

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ $\sum\left|\frac{(-1)^{n+1}}{n^2}\right| = \sum\frac{1}{n^2}$ p-series conv; $\sum\frac{(-1)^{n+1}}{n^2}$ alt. series conv

c. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ alt. harmonic series conv. cond.

d. $\sum_{n=1}^{\infty}(-1)^{n+1}3^n$ diverges

e. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln n}$ bad question! not defined for n=1; if n starts at 2, the series is cond. conv.

12. Which of the following series is conditionally convergent?

a. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$ **b.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$ **c.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}$ **d.** $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}$ **e.** $\sum_{n=1}^{\infty}(-1)^{n+1}e^n$

13. $\sum_{n=0}^{\infty}\left(\frac{2}{5}\right)^n = \frac{1}{1-\frac{2}{5}} = \frac{1}{\frac{3}{5}} = \frac{5}{3}$

a. $\frac{2}{5}$ **b.** $\frac{2}{3}$ **c.** 1 **d.** $\frac{5}{3}$ **e.** $\frac{5}{2}$

14. If $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, which of the following series also diverges?

a. $\sum_{n=1}^{\infty}\frac{1}{n^2}$ p-series p=2 conv

b. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ alt. series conv

c. $\sum_{n=1}^{\infty}\frac{0.5}{n} = 0.5\sum\frac{1}{n}$ constant * div. series

d. $\sum_{n=1}^{\infty}\frac{10^6}{n^2}$ p-series p=2

e. $\sum_{n=1}^{\infty}\frac{1}{n^n}$ root test conv

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15. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^p}$. (DO NOT USE A CALCULATOR)

a. Determine the values of p for which the series converges. Justify your conclusion.

alt series test $\lim_{n\to\infty} \frac{1}{n^p}$ must $=0$ to converge

that happens if p is positive $\Rightarrow p>0$

b. Find the maximum value of the error if k terms are used to approximate the sum of the series.

if k terms are used, the next omitted term is the k+1 term

$$\text{error} = \left|\frac{(-1)^{k+1+1}}{(k+1)^p}\right| = \frac{1}{(k+1)^p}$$

PROBEMS 16-17 REQUIRE THE USE OF A CALCULATOR.

16. a. Find the sum $\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n^n}$.
$\frac{1}{1^1} + \frac{-1}{2^2} + \frac{1}{3^3} + \frac{-1}{4^4} + \frac{1}{5^5} + \frac{-1}{6^6} + \frac{1}{7^7} + \frac{-1}{8^8} + \frac{1}{9^9} + \frac{-1}{10^{10}}$

$\approx .783$

b. What is the maximum value of the error if the sum $\sum_{n=1}^{10} \frac{(-1)^{n+1}}{n^n}$ is approximated by the answer in part (a)?

$$|a_{11}| = \left|\frac{1}{11^{11}}\right| = 3.5049 \times 10^{-12}$$
$$\approx .0000000000035049$$

c. If the sum of the first twenty terms was used instead in part (a), would the error in part (b) be increased or decreased? Explain your answer.

$|a_{21}| = 1.71157 \times 10^{-28}$ error is decreased since a_n is decreasing

17. **a.** Find the error when the alternating series $\sum_{n=0}^{\infty} (-0.25)^n$ is approximated by the first four terms of the series.

$\underbrace{1 + -.25 + .0625 + -.015625}_{.796875} + \ldots$

$\text{error} = \left|(-0.25)^4\right| = \frac{1}{256} \approx .004$

b. Find the exact sum of the series and use it to find the actual error between the exact sum and the approximate sum using the first four terms of the series. Compare your answer to that in part (a).

geometric $r = -.25$ $a_1 = 1$

$\text{sum} = \frac{1}{1 - -.25} = .8$

actual error $= |\text{exact sum} - \text{approx. sum}|$

$|.8 - .796875|$

$= .003125$ -- less than part a

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AP Calculus BC REVIEW TAYLOR and MACLAURIN SERIES

Multiple Choice: Calculator-active

1. The coefficient of x^3 in the Taylor series for $f(x) = \ln x$ centered about $x = 1$ is

a. $\frac{1}{6}$ b. $\frac{2}{3}$ c. $\frac{1}{2}$ d. $\frac{1}{3}$ e. $\frac{1}{4}$

		at $x=1$
$f(x)$	$\ln x$	0
$f'(x)$	$\frac{1}{x} = x^{-1}$	1
$f''(x)$	$-x^{-2}$	-1
$f'''(x)$	$2x^{-3}$	2

cubic term $\frac{2}{3!}(x-1)^3$

$\frac{2}{6}(x-1)^3$

2. What are all values of x for which the series $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n}$ converges?

a. $-1 \le x < 1$ b. $-1 \le x \le 1$ c. $0 < x < 2$ d. $0 \le x < 2$ e. $0 \le x \le 2$

ratio test: $\lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{n}{n+1}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\right| = 1\cdot|(x-1)^1|$ conv. if $|x-1|<1$

$-1 < x-1 < 1$

$0 < x < 2$

check endpoints:

$x=0$ $\sum_1^{\infty}\frac{(-1)^n}{n}$ conv alt. series test

$x=2$ $\sum_1^{\infty}\frac{(1)^n}{n}$ div p-series $p=1$

3. If $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$, then $\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{n!}$ must be

$\to = \sum_1^{\infty}\frac{(-1)^n(x^2)^n}{n!} = \sum_1^{\infty}\frac{(-x^2)^n}{n!}$

need to subtract the $n=0$ term since bounds changed

so "x" is replaced by x^2

a. e^{-x^2} b. $\cos x$ c. $1-e^{-x^2}$ d. $e^{-x^2}-1$ e. None of these

Free Response: A Graphing Calculator may be used.

4. Consider the power series $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$. Let f be the function given by $f(x) = e^{-\frac{x^2}{3}}$.

a. Find $p(x)$, the fourth degree Maclaurin Polynomial for $f(x)$.

$1 + \frac{-x^2}{3} + \frac{\left(\frac{-x^2}{3}\right)^2}{2} = 1 - \frac{1}{3}x^2 + \frac{1}{18}x^4$

b. Write $f(x)$ as a power series using summation notation.

$\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{3^n \cdot n!}$

$1 - \frac{1}{3}x^2 + \frac{1}{18}x^4 - \frac{x^6}{27\cdot 3!} + \frac{x^8}{81\cdot 4!} - \cdots$

$3^0\cdot 0!$ $3^1\cdot 1!$ $3^2\cdot 2!$ 3^3

c. Use the polynomial $p(x)$ in part (a) to approximate $f(0.3)$ to 5 decimal places.

$f(0.3) \approx 1 - \frac{1}{3}(.3)^2 + \frac{1}{18}(.3)^4 = .97045$

d. Find the interval of convergence of the power series for $f(x)$ about $x = 0$. Show the analysis that leads to your conclusion.

$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{3^{n+1}(n+1)!} \div \frac{(-1)^n x^{2n}}{3^n\cdot n!}\right| = \lim_{n\to\infty}\left|(-1)^1\cdot 3^{-1}\cdot\frac{n!}{(n+1)!}\cdot x^2\right| = \frac{1}{3}\cdot 0\cdot|x^2| = 0 < 1$

$\frac{n!}{(n+1)!} \to \frac{1}{n}$

converges for all x

I.O.C. $(-\infty, \infty)$

e. Show that $|f(x) - p(x)| < 0.0001$ for $-0.3 \le x \le 0.3$.

See part ~~c~~ b -- alt. series next omitted term

$\left|-\frac{x^6}{27\cdot 3!}\right|\Bigg|_{x=.3} = 4.5\times 10^{-6} \approx .0000045$

less than .0001

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Multiple Choice: No Calculator

5. For $-1 < x \le 1$ if $f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{3n-2}}{3n-2}$ then $f'(x)=$ $\quad \sum \frac{(-1)^{n+1}}{3n-2}\cdot(3n-2)x^{3n-2-1}$

a. $f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}x^{3n}$ **b.** $f(x)=\sum_{n=1}^{\infty}(-1)^{n+1}x^{3n-3}$ **c.** $f(x)=\sum_{n=1}^{\infty}(-1)^{3n}x^{3n}$

d. $f(x)=\sum_{n=1}^{\infty}(-1)^{n}x^{3n-3}$ **e.** $f(x)=\sum_{n=1}^{\infty}(-1)^{n}x^{3n}$

6. $\sin(2x)=$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$\sin(2x) = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \ldots$

a. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots+\frac{(-1)^{n-1}x^{2n-1}}{(2n-1)!}+\ldots$

b. $(2x)-\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}-\frac{(2x)^7}{7!}+\ldots+\frac{(-1)^{n-1}(2x)^{2n-1}}{(2n-1)!}+\ldots$

c. $\frac{(2x)^2}{2}+\frac{(2x)^4}{4!}+\frac{(2x)^6}{6!}+\ldots+\frac{(2x)^{2n}}{(2n)!}+\ldots$

d. $\frac{x^2}{2}+\frac{x^4}{4!}+\frac{x^6}{6!}+\ldots+\frac{x^{2n}}{(2n)!}+\ldots$

e. $(2x)+\frac{(2x)^3}{3!}+\frac{(2x)^5}{5!}+\frac{(2x)^7}{7!}+\ldots+\frac{(2x)^{2n-1}}{(2n-1)!}+\ldots$

Free Response: No Calculator

$5+(x-3)-\frac{2(x-3)^2}{4}+\frac{8(x-3)^3}{9}$

7. Consider the power series $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n 2^n (x-3)^n}{n+1}$. $\rightarrow \left(\frac{5}{2},\frac{7}{2}\right]$

a. For what values of x does the series converge? Show the analysis that leads to your conclusion.

$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}(x-3)^{n+1}}{n+1+1} \div \frac{(-1)^n 2^n (x-3)^n}{n+1}\right| = \lim_{n\to\infty}\left|(-1)^1 \cdot 2^1 \cdot \frac{n+1}{n+2}\cdot(x-3)\right| = |2(x-3)|$ $\quad |2(x-3)|<1$ $\quad -1<2x-6<1$ $\quad 5<2x<7$ $\quad \frac{5}{2}<x<\frac{7}{2}$

b. Let g be a function satisfying $g(3)=5$ and $g'(x)=f(x)$ for all x. Write the first four terms and the general term of the Taylor series for g about $x=3$.

$g(x)=\int g'(x)dx = \int f(x)dx = \sum\left[\frac{(-1)^n 2^n}{n+1}\cdot\frac{(x-3)^{n+1}}{n+1}\right]+C$ $\quad 5=\sum\frac{(-1)^n 2^n (0)^{n+1}}{(n+1)^2}+C$ $\quad C=5$

check endpts: $x=\frac{5}{2}$ $\sum\frac{(-1)^n 2^n(-\frac{1}{2})^n}{n+1}$

c. What is the interval of convergence of the power series for g found in part b? Show the analysis that leads to your conclusion.

$\sum\frac{(-1)^{2n} 2^n}{2^n(n+1)}$

recheck endpts: $x=\frac{5}{2}$ $5+\sum\frac{(-1)^n 2^n(-\frac{1}{2})^{n+1}}{(n+1)^2} = \sum\frac{(-1)^{2n+1}(2^{-1})}{(n+1)^2}$ converges $\left[\frac{5}{2},\frac{7}{2}\right]$

$\sum\frac{1}{n+1}$ div

$x=\frac{7}{2}$ $5+\sum\frac{(-1)^n 2^n(\frac{1}{2})^{n+1}}{(n+1)^2} = \sum\frac{(-1)^n(2)^{-1}}{(n+1)^2}$ converges

$x=\frac{7}{2}$ $\sum\frac{(-1)^n 2^n(\frac{1}{2})^n}{n+1}$

8. Consider $f(x)=\frac{4}{3x+2}$.

$\sum\frac{(-1)^n}{n+1}$ conv alt. smc

a. Find a power series for $f(x)$ centered at $x=3$.

b. Find the interval of convergence for your power series found in part **(a)**. next page

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a) $f(x) = \dfrac{4}{3x+2}$

Similar to geom. form $\dfrac{a}{1-r}$ (sum)

$$\frac{4}{3x+2} = \frac{4}{\underbrace{3x-9}_{\substack{3(x-3)\\ \uparrow \\ \text{center at } 3}}+2+9} = \frac{4}{3(x-3)+11} = \frac{\frac{4}{11}}{\frac{3(x-3)+11}{11}} = \frac{\frac{4}{11}}{1 - -\frac{3}{11}(x-3)}$$

$a_1 = \frac{4}{11}$

$r = -\frac{3}{11}(x-3)$

$$\sum_{n=0}^{\infty} \frac{4}{11}\left(-\frac{3}{11}(x-3)\right)^n = \sum_{n=0}^{\infty} \frac{4}{11} \cdot \frac{(-3)^n (x-3)^n}{11^n} = 4\sum_{n=0}^{\infty} \frac{(-3)^n (x-3)^n}{11^{n+1}}$$

b) geom. series converges if $|r| < 1$

$$\left| -\frac{3}{11}(x-3) \right| < 1$$

$$-1 < -\frac{3}{11}(x-3) < 1$$

$$\frac{11}{3} > x - 3 > \frac{11}{3}$$

$$\frac{20}{3} > x > -\frac{2}{3}$$

$$\left(-\frac{2}{3}, \frac{20}{3}\right)$$