Taylor & Maclaurin Polynomials

Goal: given a function, find a polynomial that agrees with the function on some interval.

Review: Given $f(x)$. Find the eqn. of the tangent line to $f(x)$ at $x = c$.

$$\text{slope} = f'(x)\Big|_{x=c} = f'(c)$$

point $(c, f(c))$

$$y - f(c) = f'(c)(x-c)$$

$$y = \underbrace{f(c) + f'(c)(x-c)}_{\underbrace{\text{linear}}_{1^{st}\text{ deg.}}\text{ polynomial}}$$

n^{th} Taylor Polynomial at $x = c$ ← center

$$P_n(x) = \frac{f(c)}{0!} + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

n ← degree

If $c = 0$, then the polynomial is called a "Maclaurin" polynomial.

EX1 $f(x)=e^x$. Write a 1^{st} degree, 3^{rd} degree, and a 6^{th} degree polynomial if $c=0$.

		at $\overset{x}{c}=0$
$f(x)$	e^x	$e^0=1$
$f'(x)$	e^x	1
$f''(x)$	e^x	1
$f'''(x)$	e^x	1
$f^{(4)}(x)$	e^x	1
$f^{(5)}(x)$	e^x	1
$f^{(6)}(x)$	e^x	1

$$P_1(x) = 1+\frac{1}{1!}(x-0)^1 = 1+x$$

$$P_3(x) = 1+x+\frac{1}{2!}(x-0)^2+\frac{1}{3!}(x-0)^3$$

$$= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3$$

$$P_6(x) = 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{4!}(x-0)^4+\frac{1}{5!}(x-0)^5$$

$$+\frac{1}{6!}(x-0)^6$$

$$= 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5$$

$$+\frac{1}{720}x^6$$

EX2 $f(x)=\cos x$. Write the $\underbrace{\text{Maclaurin}}_{c=0}$ polynomials P_0, P_2, P_4, and P_6.

		at $c=0$
$f(x)$	$\cos x$	1
$f'(x)$	$-\sin x$	0
$f''(x)$	$-\cos x$	-1
$f'''(x)$	$\sin x$	0
$f^{(4)}(x)$	$\cos x$	1
$f^{(5)}(x)$	$-\sin x$	0
$f^{(6)}(x)$	$-\cos x$	-1

$$P_0 = 1$$

$$P_2 = 1+\frac{0}{1!}(x-0)^1+\frac{-1}{2!}(x-0)^2 = 1-\frac{1}{2!}x^2$$

$$P_4 = 1-\frac{1}{2!}x^2+\frac{0}{3!}(x-0)^3+\frac{1}{4!}(x-0)^4$$

$$= 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4$$

$$P_6 = 1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6$$

EX3 $f(x)=x^2e^{-x}$. Write a 4^{th} degree Maclaurin polynomial.

$c=0$

$$f'(x)=x^2\cdot -e^{-x}+e^{-x}\cdot 2x = -e^{-x}(x^2-2x)$$

$$f''(x)= -e^{-x}(2x-2)+(x^2-2x)\cdot e^{-x}$$

$$= e^{-x}(-2x+2+x^2-2x) = e^{-x}(x^2-4x+2)$$

$$f'''(x)= e^{-x}(2x-4)+(x^2-4x+2)(-e^{-x})$$

$$= -e^{-x}[-2x+4+x^2-4x+2] = -e^{-x}(x^2-6x+6)$$

$$f^{(4)}(x)= -e^{-x}(2x-6)+(x^2-6x+6)(e^{-x})$$

$$= e^{-x}[-2x+6+x^2-6x+6]$$

$$= e^{-x}(x^2-8x+12)$$

$$f(0)= 0^2\cdot e^{-0}=0$$

$$f'(0)= -e^{-0}(0^2-2\cdot 0)=0$$

$$f''(0)= e^{-0}(0^2-4\cdot 0+2)=2$$

$$f'''(0)= -1(6)=-6$$

$$f^{(4)}(0)= 1(12)=12$$

$$P_4 = 0+0\cdot x^1+\frac{2}{2!}\cdot x^2+\frac{-6}{3!}x^3+\frac{12}{4!}x^4$$

$$= x^2-x^3+\frac{1}{2}x^4$$

EX4 $f(x) = \sin x$. Write a 3rd deg. Taylor polyn. Centered at $\pi/6$.

		at $c = \pi/6$
$f(x)$	$\sin x$	$\frac{1}{2}$
$f'(x)$	$\cos x$	$\frac{\sqrt{3}}{2}$
$f''(x)$	$-\sin x$	$-\frac{1}{2}$
$f'''(x)$	$-\cos x$	$-\frac{\sqrt{3}}{2}$

$$P_3(x) = \frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{1!}\left(x - \frac{\pi}{6}\right)^1 + \frac{-\frac{1}{2}}{2!}\left(x - \frac{\pi}{6}\right)^2 + \frac{-\frac{\sqrt{3}}{2}}{3!}\left(x - \frac{\pi}{6}\right)^3$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$

EX5 $f(x) = \sqrt[3]{x}$. Write the 3rd deg. Taylor polynomial if $c = 8$.

		at $c=8$
$f(x)$	$x^{1/3}$	2
$f'(x)$	$\frac{1}{3}x^{-2/3}$	$\frac{1}{12}$
$f''(x)$	$-\frac{2}{9}x^{-\frac{5}{3}}$	$-\frac{2}{9} \cdot \frac{1}{32} = -\frac{1}{144}$
$f'''(x)$	$\frac{10}{27}x^{-\frac{8}{3}}$	$\frac{10}{27} \cdot \frac{1}{256} = \frac{5}{3456}$

$$P_3(x) = 2 + \frac{\frac{1}{12}}{1!}(x-8)^1 - \frac{\frac{1}{144}}{2!}(x-8)^2 + \frac{\frac{5}{3456}}{3!}(x-8)^3$$