AP Calculus AB Unit #3 Notes
Applications of Derivatives: Important Theorems

Mean Value Theorem

If the function f is continuous on the closed interval [a, b] and differentiable on the open interval (a, b), then there exists at least one number c in the open interval (a, b) such that:

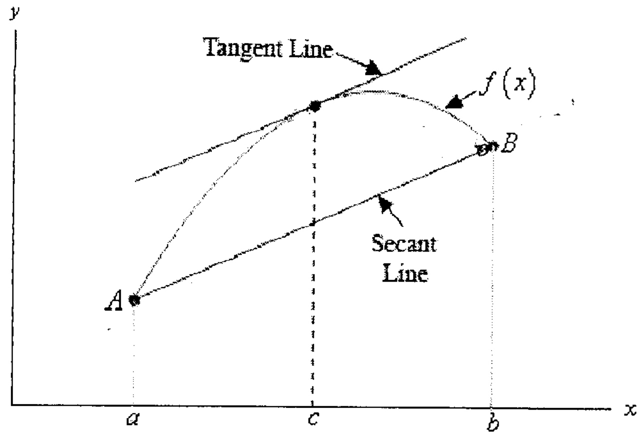
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

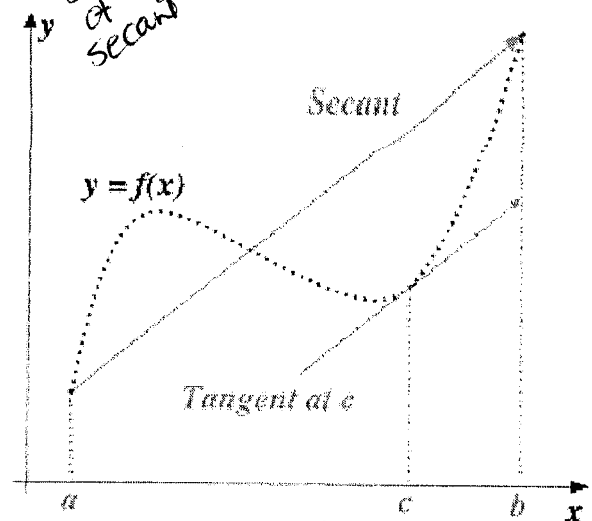
$(a, f(a))$ $(b, f(b))$

x-value

slope of tangent

slope of secant





Example #1:

If the function f is defined on [1, 3] by f(x) = 4 – 3/x, show that the MVT can be applied to f and find a number c which satisfies the conclusion.

$\rightarrow -3x^{-1}$

$\rightarrow f'(x) = 3x^{-2} = \frac{3}{x^2}$

① f is cont. on [1, 3] ✓
② f is diff. on (1, 3) ✓
MVT applies

$(1, 1)$ $(3, 3)$

$\frac{3}{x^2} = \frac{3-1}{3-1} = 1$

$x^2 = 3$

$x = \pm\sqrt{3}$

$\boxed{c = \sqrt{3}}$

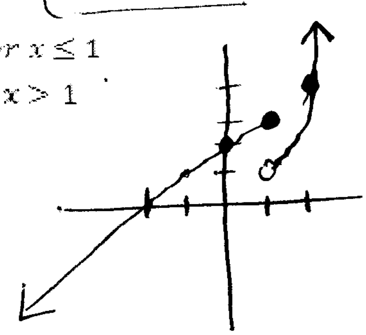
Example #2:

Sketch a graph of the function f if $f(x) = \begin{cases} x+2, & for\ x \le 1 \\ x^2, & for\ x > 1 \end{cases}$

Show that f fails to satisfy the MVT on the interval [-2, 2].

① f is NOT cont at x = 1

② f is NOT diff at x = 1



Example #3:

Suppose that s(t) = t² – t + 4 is the position of the motion of a particle moving along a line.

$s'(t) = 2t - 1$

a) Explain why the function s satisfies the hypothesis of the MVT.

① s(t) is cont. everywhere
② s(t) is diff. everywhere

b) Find the value of t in [0, 3] where instantaneous velocity is equal to the average velocity.

$(0, 4)$
12 $(3, 10)$

$2t - 1 = \frac{10-4}{3-0} = 2$

$2t = 3$

$t = 3/2$

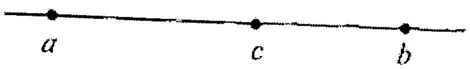
Rolle's Theorem

zero

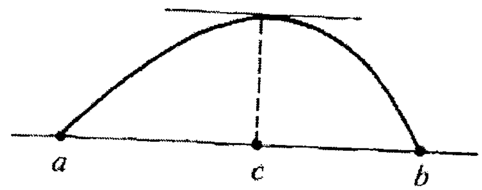
Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval [a, b]
2. f is differentiable on the open interval (a, b)
3. f(a) = f(b)

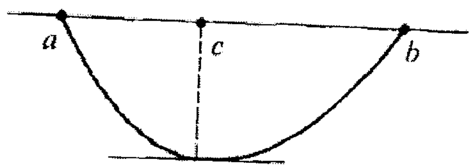
Then there is a number c in (a, b) such that f′(c) = 0.



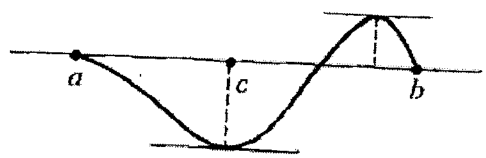
Case 1, $m = M = 0$



Case 2, $M > 0$



Case 3, $m < 0$



Case 2 and 3, $m < 0 < M$

Example #4:

Verify the function satisfies the three hypotheses of Rolle's Theorem on the given interval. Then find all numbers c that satisfy the conclusion of Rolle's Theorem.

a) $f(x) = x^2 - 4x + 1$, [0, 4]

① f is cont on [0, 4] ✓
② f is diff on (0, 4) ✓
③ $f(0) = f(4)$ ✓ Rolle's Thm applies
$1 \qquad 16-16+1 = 1$

$2x - 4 = 0$
$x = 2$

$\boxed{c = 2}$

b) $f(x) = \sin(2\pi x)$, [-1, 1]

① f is cont. on [-1, 1] ✓
② f is diff. on (-1, 1) ✓
③ $f(-1) = f(1)$ $\quad \sin(-2\pi) = 0 \quad \sin(2\pi) = 0$ ✓

Rolle's thm applies

$-1 < x < 1$
$-2\pi < 2\pi x < 2\pi$

$2\pi \cos(2\pi x) = 0$
$\cos(2\pi x) = 0$
$2\pi x = \cos^{-1}(0)$
$2\pi x = \frac{\pi}{2}, \frac{3\pi}{2}, -\frac{\pi}{2}, -\frac{3\pi}{2}$
$x = \frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{3}{4}$

Example #5:

Let $f(x) = 1 - x^{2/3}$. Show that f(-1) = f(1) but there is no number c in (-1, 1) such that f′(c) = 0. Why does this contradict Rolle's Theorem?

$f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0$ ✓
$f(1) = 1 - (1)^{2/3} = 1 - 1 = 0$

$f'(x) = -\frac{2}{3}x^{-1/3}$

$\frac{-2}{3\sqrt[3]{x}} = 0$

no soln.

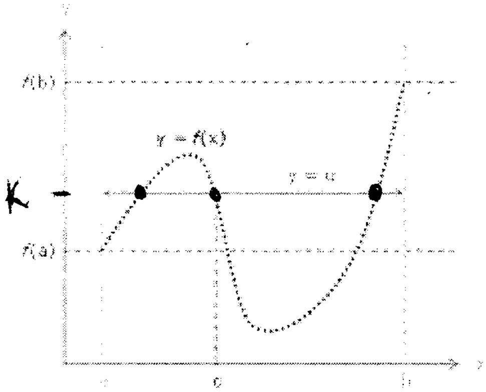
f(x) is not diff at x=0
⇒ f is not diff on (-1, 1)

y-value

Intermediate Value Theorem

If f is a continuous function on the closed interval [a, b], with f(a) ≠ f(b), and k is a number between f(a) and f(b), then there exists at least one number c in (a, b) for which f(c) = k.

x-value



x-intercept
y-value = 0

Example #6

Use the Intermediate Value Theorem to show that there is a zero for the given function in the specified interval.

a) $f(x) = x^3 - 3x + 1$, [0, 1]

① f(x) is cont [0, 1] ✓
② $f(0) \neq f(1)$ ✓
$\quad 1 \quad\quad -1$
③ $-1 < 0 < 1$ ✓

IVT applies

Because IVT applies, there is an x-value, "c", in (0,1) where f(c) = 0

b) $g(x) = \ln(x) - e^{-x}$, [1, 2]

① g(x) is cont [1,2] ✓
② $g(1) \neq g(2)$ ✓
$g(1) = \ln 1 - e^{-1} = -.368$
$g(2) = \ln 2 - e^{-2} = .558$
③ $-.368 < 0 < .558$ ✓

IVT applies

Because IVT applies, there is an x-value, "c", in (1,2) where f(c) = 0.