NOTES on Vector and Parametric free response questions

Things to know about vectors . . .

vector—a directed line segment that has an initial point and a terminal point
component form: $\vec{v}$ or $\mathbf{v} = \langle a, b \rangle$
linear combination form: $\mathbf{v} = ai + bj$
magnitude = the length of a vector = $\lvert \mathbf{v} \rvert = \lVert \vec{v} \rVert = \sqrt{(horizontal\ component)^2 + (vertical\ component)^2}$
speed = $\lvert v(t) \rvert$ = the magnitude of the velocity components
direction (a unit vector) = $\frac{velocity\ vector}{speed} = \frac{v(t)}{\lvert v(t) \rvert}$

note: vector/parametric free response questions are usually calculator-active

Example 1

A particle moves in the xy-plane so that its position at any time t, for $-\pi \le t \le \pi$, is given by $x(t) = \sin(3t)$ and $y(t) = 2t$.

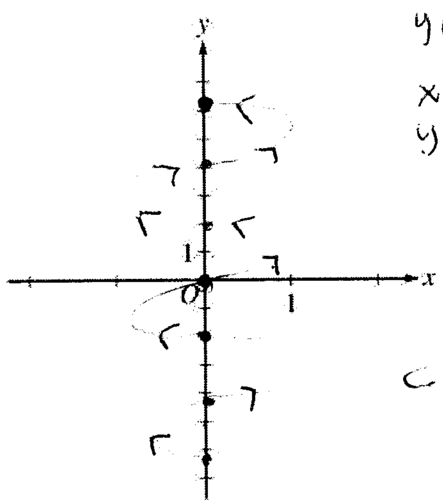
(a) Sketch the path of the particle in the xy-plane provided. Indicate the direction of motion along the path.

(b) Find the range of $x(t)$ and the range of $y(t)$.

(c) Find the smallest positive value of t for which the x-coordinate of the particle is a local maximum. What is the speed of the particle at this time?

(d) Is the distance traveled by the particle from $t = -\pi$ to $t = \pi$ greater than 5π? Justify your answer.

arc length



a. $x(-\pi) = 0$, $y(-\pi) = -2\pi$ start

$x(\pi) = 0$, $y(\pi) = 2\pi$ end

b. $-1 \le x(t) \le 1$

$-2\pi \le y(t) \le 2\pi$

c. $x(t) = \sin(3t)$

$x'(t) = 3\cos(3t)$

$3\cos(3t) = 0$

$\cos(3t) = 0$

$3t = \cos^{-1}(0)$

$3t = \pi/2$

$t = \pi/6$

$y'(t) = 2$

speed $= \sqrt{(3\cos(3t))^2 + (2)^2}\Big|_{t=\pi/6}$

$= 2$

d. $\int_{-\pi}^{\pi} \sqrt{(3\cos(3t))^2 + (2)^2}\, dt$

17.973

$5\pi \approx 15.708$ yes

Example 2

A moving particle has position $(x(t), y(t))$ at time t. The position of the particle at time $t = 1$ is (2,6) and the velocity vector at any time $t > 0$ is given by $\left(1 - \frac{1}{t^2}, 2 + \frac{1}{t^2}\right)$.

(a) Find the acceleration vector at time $t = 3$. $-1t^{-2}$ t^{-2}

(b) Find the position of the particle at time $t = 3$.

(c) For what time $t > 0$ does the line tangent to the path of the particle at $(x(t), y(t))$ have a slope of 8?

(d) The particle approaches a line as $t \to \infty$. Find the slope of this line. Show the work that leads to your conclusion.

a) $a(t) = \left\langle 2t^{-3}, -2t^{-3} \right\rangle = \left\langle \frac{2}{t^3}, \frac{-2}{t^3} \right\rangle \Big|_{t=3} = \left\langle \frac{2}{27}, \frac{-2}{27} \right\rangle$

b) $s(t) = \langle t + t^{-1} + C, \; 2t - t^{-1} + C \rangle$

position $t = 1$ (2,6)

$1 + \frac{1}{1} + C = 2$ $\quad\quad 2 - \frac{1}{1} + C = 6$

$C = 0$ $\quad\quad C = 5$

$s(t) = \left\langle t + \frac{1}{t}, \; 2t - \frac{1}{t} + 5 \right\rangle \Big|_{t=3}$

$\left\langle \frac{10}{3}, \frac{32}{3} \right\rangle$

c) slope $= \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 8$

$2 + \frac{1}{t^2} = 8\left(1 - \frac{1}{t^2}\right)$

$2t^2 + 1 = 8t^2 - 8$

$9 = 6t^2$

$\frac{9}{6} = t^2$

$t = \sqrt{\frac{9}{6}} = \frac{3}{\sqrt{6}}$

4 d) $\lim_{t \to \infty} \frac{2 + \frac{1}{t^2}}{1 - \frac{1}{t^2}} = 2$