#1

1977 AB5

a. $v(t) = \int \left(2t - 10 + \frac{12}{t}\right) dt = t^2 - 10t + 12\ln|t| + C$

$9 = 1^2 - 10(1) + 12\ln 1 + C$

$9 = 1 - 10 + 0 + C$

$18 = C$

$\boxed{v(t) = t^2 - 10t + 12\ln|t| + 18}$

b. $v'(t) = 2t - 10 + 12 \cdot \frac{1}{t} = \frac{2t^2 - 10t + 12}{t}$

undef at $t = 0$

$2t^2 - 10t + 12 = 0$

$2(t^2 - 5t + 6) = 0$

$2(t-3)(t-2) = 0$

$t = 3, 2$

$+ \quad - \quad +$

$0 \quad 2 \quad 3$

$\boxed{t = 2}$ because the velocity changes from increasing to decreasing.

c. $x(t) = \int \left(t^2 - 10t + 12\ln t + 18\right) dt$

$= \frac{1}{3}t^3 - 5t^2 + 12(t\ln t - t) + 18t + C$

$-16 = \frac{1}{3} - 5 + 12(0 - 1) + 18 + C$

$-16 = \frac{1}{3} - 5 - 12 + 18 + C$

$-\frac{52}{3} = C$

$\boxed{x(t) = \frac{1}{3}t^3 - 5t^2 + 12t\ln t + 6t - \frac{52}{3}}$

$\int \ln x \, dx = x\ln x - x + C$

$x(\ln x - 1) + C$

#2

1977 BC 7

a. $F(0) = \int_0^0 \frac{1}{1+t^4}\,dt = \boxed{0}$

b. $F'(x) = \frac{d}{dx}\int_0^x \frac{1}{1+t^4}\,dt = \frac{1}{1+x^4}$

$F'(1) = \frac{1}{1+(1)^4} = \boxed{\frac{1}{2}}$

c. $F(3) - F(1) = \int_1^3 \frac{1}{1+x^4}\,dx$

For $1 \le t \le 3$: $\left.\frac{1}{1+t^4}\right|_1 = \frac{1}{2}$ $\quad \left.\frac{1}{1+t^4}\right|_3 = \frac{1}{1+81} = \frac{1}{82}$

So $\frac{1}{1+x^4} \le \frac{1}{2}$

$\int_1^3 \frac{1}{2}\,dt = \frac{1}{2}t + c\Big|_1^3 = \frac{3}{2} + c - \left(\frac{1}{2} + c\right) = 1$

d. $F(x) + F(-x) = 0$

$\int_{-x}^0 \frac{1}{1+t^4}\,dt = \int_0^x \frac{1}{1+t^4}\,dt$ b/c $\frac{1}{1+t^4}$ is even

$F(x) + F(-x) = \int_0^x \frac{1}{1+t^4}\,dt + \int_0^{-x} \frac{1}{1+t^4}\,dt$

$= \int_0^x \frac{1}{1+t^4}\,dt - \int_{-x}^0 \frac{1}{1+t^4}\,dt$

same quantities since even

$= \boxed{0}$

#3

1981 AB7

a. $\frac{1}{3-1}\int_1^3 f(x)\,dx = \frac{1}{2}\cdot\frac{5}{2} = \boxed{\frac{5}{4}}$

b. $\int_3^5 (2f(x)+6)\,dx = 2\int_3^5 f(x)\,dx + \int_3^5 6\,dx$

$= 2\left[\int_1^5 f(x)dx - \int_1^3 f(x)dx\right] + (6x+C)\Big|_3^5$

$= 2\left[\,10 - \frac{5}{2}\right] + \quad 30 + C - (18+C)$

$= \boxed{27}$

c. $\int_1^3 f(x)dx = \frac{5}{2}$

$\int_1^3 (ax+b)dx = \frac{5}{2}$

$\frac{a}{2}x^2 + bx + C\Big|_1^3 = \frac{5}{2}$

$\frac{9}{2}a + 3b + C - \left(\frac{a}{2} + b + C\right) = \frac{5}{2}$

$4a + 2b = \frac{5}{2}$

$8a + 4b = 5$

$\int_1^5 f(x)\,dx = 10$

$\int_1^5 (ax+b)dx = 10$

$\frac{a}{2}x^2 + bx + c\Big|_1^5 = 10$

$\frac{25}{2}a + 5b + c - \left(\frac{a}{2} + b + c\right) = 10$

$12a + 4b = 10$

$8a + 4b = 5$

$-1(12a + 4b = 10)$

$8a + 4b = 5$

$-12a - 4b = -10$

$-4a = -5$

$\boxed{a = \frac{5}{4}}$

$8\left(\frac{5}{4}\right) + 4b = 5$

$10 + 4b = 5$

$4b = -5$

$\boxed{b = -\frac{5}{4}}$

1977 AB5/BC3
Solution

(a) $v(t) = t^2 - 10t + 12\ln t + C$

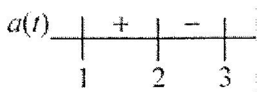
$9 = v(1) = 1 - 10 + 12(0) + C$

$C = 18$

$$\boxed{v(t) = t^2 - 10t + 12\ln t + 18}$$

(b) $a(t) = \dfrac{2t^2 - 10t + 12}{t} = \dfrac{2(t-2)(t-3)}{t}$

$a(t) = 0$ when $t = 2$ and $t = 3$.



Since the velocity is increasing for $1 \le t < 2$ and decreasing for $2 < t < 3$, the velocity is a maximum at $\boxed{t = 2.}$

or

The maximum will be at $t = 2$ or at an endpoint.

$v''(t) = a'(t) = \dfrac{2t^2 - 12}{t^2}$.

Since $v'(3) = 0$ and $v''(3) = \dfrac{2}{3} > 0$, then v has a relative minimum at $t = 3$.

$v(1) = 9 < v(2) = 2 + 12\ln 2 \approx 2 + 12\cdot(0.7) = 10.4$

Therefore the maximum velocity is at $t = 2$.

(c) $x(t) = \int v(t)\,dt = \dfrac{t^3}{3} - 5t^2 + 18t + 12(t\ln t - t) + D$

$-16 = \dfrac{1}{3} - 5 + 18 - 12 + D$

$D = -16 - \dfrac{4}{3} = -\dfrac{52}{3}$

$$\boxed{x(t) = \frac{t^3}{3} - 5t^2 + 6t + 12t\ln t - \frac{52}{3}}$$

6. 1977 BC7

Let $F(x)=\int_0^x \frac{1}{1+t^4}dt$ for all real numbers x.

(a) Find F(0).
(b) Find F′(1).
(c) Justify that F(3) – F(1) < 1.
(d) Justify that F(x) + F(–x) = 0 for all real numbers x.

a.) $F(0)=\int_0^0 \frac{dt}{1+t^4} = 0$

b.) $F'(x)=\frac{1}{1+x^4} \Rightarrow F'(1)=\frac{1}{2}$

c.) $F(3)-F(1)=\int_0^3 \frac{dt}{1+t^4} - \int_0^1 \frac{dt}{1+t^4} = \int_1^3 \frac{dt}{1+t^4}$

$\frac{1}{1+t^4}$ decreasing on $[1,3] \Rightarrow \frac{1}{1+t^4} \le \frac{1}{2}$ and

$\int_1^3 \frac{dt}{1+t^4} < \int_1^3 \frac{1}{2}\,dt = 1.$

d.) $F(-x)=\int_0^{-x} \frac{1}{1+t^4}\,dt$

Let $t=-u$. Then $dt=-du$ and

$F(-x) = -\int_0^x \frac{1}{1+u^4}\,du = -F(x)$

Therefore $F(x)+F(-x)=F(x)-F(x)=0$

OR

$\frac{1}{1+t^4}$ is even $\Rightarrow F(x)=\int_0^x \frac{dt}{1+t^4} = \int_{-x}^0 \frac{dt}{1+t^4} = -\int_0^{-x} \frac{dt}{1+t^4} = -F(-x)$

Then $F(x)+F(-x) = -F(-x)+F(-x)=0$

1981 AB7
Solution

(a) Average (mean) value $= \dfrac{1}{3-1}\displaystyle\int_1^3 f(x)\,dx = \dfrac{1}{2}\left(\dfrac{5}{2}\right) = \boxed{\dfrac{5}{4}}$

(b) $\displaystyle\int_3^5 \left(2f(x)+6\right)dx$

$\displaystyle = 2\int_3^5 f(x)\,dx + \int_3^5 6\,dx$

$\displaystyle = 2\left(\int_1^5 f(x)\,dx - \int_1^3 f(x)\,dx\right) + \int_3^5 6\,dx$

$\displaystyle = 2\left(10 - \frac{5}{2}\right) + 6(5-3)$ 12

$= 15 + 12 = \boxed{27}$

(c) $\displaystyle \frac{5}{2} = \int_1^3 (ax+b)\,dx = \left(\frac{ax^2}{2} + bx\right)\Bigg|_1^3 = 4a + 2b$

$\displaystyle 10 = \int_1^5 (ax+b)\,dx = \left(\frac{ax^2}{2} + bx\right)\Bigg|_1^5 = 12a + 4b$

Solving these two simultaneous equations yields $\boxed{a = \dfrac{5}{4},\ b = -\dfrac{5}{4}}$.

$12a + 4b = 10$