AP Calculus

Practice: Integration & Inverse Trig

Find the derivative of each function.

1) $f(x) = 2\arcsin(x-1)$

$\frac{2}{\sqrt{-x^2+2x}}$

2) $g(x) = 3\arccos\frac{x}{2}$

$\frac{-3}{\sqrt{4-x^2}}$

3) $f(x) = \arctan\frac{x}{a}$

$\frac{a}{a^2+x^2}$

Evaluate.

4) $\int \frac{5}{\sqrt{9-x^2}}dx$

$5\arcsin\frac{x}{3}+C$

5) $\int_0^{\frac{1}{6}} \frac{1}{\sqrt{1-9x^2}}dx$

$\frac{\pi}{18}$

6) $\int \frac{7}{16+x^2}dx$

$\frac{7}{4}\arctan\frac{x}{4}+C$

7) $\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1+4x^2}dx$

$\frac{\pi}{6}$

8) $\int \frac{1}{x\sqrt{4x^2-1}}dx$

$\operatorname{arcsec}|2x|+C$

9) $\int \frac{x^3}{x^2+1}dx$

$\frac{1}{2}x^2-\frac{1}{2}\ln(x^2+1)+C$

10) $\int \frac{1}{\sqrt{1-(x+1)^2}}dx$

$\arcsin(x+1)+C$

11) $\int \frac{e^{2x}}{4+e^{4x}}dx$

$\frac{1}{4}\arctan\frac{e^{2x}}{2}+C$

12) $\int_{\pi/2}^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

$\frac{\pi}{4}$

13) $\int_0^2 \frac{dx}{x^2-2x+2}$

$\frac{\pi}{2}$

14) $\int \frac{2x}{x^2+6x+13}dx$

$\ln|x^2+6x+13|-3\arctan\frac{x+3}{2}+C$

15) $\int \frac{x-2}{(x+1)^2+4}dx$

$\frac{1}{2}\ln|x^2+2x+5|-\frac{3}{2}\arctan\frac{x+1}{2}+C$

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Integration & Inverse Trig

1. $f(x) = 2 \arcsin(x-1)$

$$f'(x) = 2 \cdot \frac{1}{\sqrt{1-(x-1)^2}} = \frac{2}{\sqrt{1-(x^2-2x+1)}} = \boxed{\frac{2}{\sqrt{-x^2+2x}}}$$

2. $g(x) = 3 \arccos \frac{x}{2}$

$$g'(x) = 3 \cdot \frac{-\frac{1}{2}}{\sqrt{1-\left(\frac{x}{2}\right)^2}} = \frac{-\frac{3}{2}}{\sqrt{1-\frac{x^2}{4}}} = \frac{-\frac{3}{2}}{\sqrt{\frac{4-x^2}{4}}} = \frac{-\frac{3}{2}}{\frac{\sqrt{4-x^2}}{2}} = \boxed{\frac{-3}{\sqrt{4-x^2}}}$$

3. $f(x) = \arctan \frac{x}{a}$

$$f'(x) = \frac{\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2} = \frac{\frac{1}{a}}{1+\frac{x^2}{a^2}} = \frac{\frac{1}{a}}{\frac{a^2+x^2}{a^2}} = \frac{1}{a} \cdot \frac{a^2}{a^2+x^2} = \boxed{\frac{a}{a^2+x^2}}$$

4. $\int \frac{5}{\sqrt{9-x^2}}\,dx$

$$5 \int \frac{du}{\sqrt{a^2-u^2}} = 5 \cdot \arcsin \frac{u}{a} + C = \boxed{5 \arcsin \frac{x}{3} + C}$$

$a^2 = 9 \quad u^2 = x^2$

$a = 3 \quad u = x$

$\frac{du}{dx} = 1 \quad du = dx$

5. $\int_0^{1/6} \frac{1}{\sqrt{1-9x^2}}\,dx$

$$\frac{1}{3} \int \frac{du}{\sqrt{a^2-u^2}} = \frac{1}{3} \arcsin \frac{u}{a} + C$$

$$= \frac{1}{3} \arcsin 3x + C \Big|_0^{1/6}$$

$$= \frac{1}{3} \arcsin \frac{1}{2} + C - \left(\frac{1}{3} \arcsin 0 + C\right)$$

$$= \frac{1}{3} \cdot \frac{\pi}{6} = \boxed{\frac{\pi}{18}}$$

$a^2 = 1 \quad u^2 = 9x^2$

$a = 1 \quad u = 3x$

$\frac{du}{dx} = 3 \quad \frac{1}{3} du = dx$

6. $\int \frac{7}{16+x^2}\,dx$

$$7 \int \frac{du}{a^2+u^2} = 7 \cdot \frac{1}{a} \arctan \frac{u}{a} + C$$

$$= \boxed{\frac{7}{4} \arctan \frac{x}{4} + C}$$

$a^2 = 16 \quad u^2 = x^2$

$a = 4 \quad u = x$

$\frac{du}{dx} = 1 \quad du = dx$

7. $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{1+4x^2}\,dx$

$a^2=1 \qquad u^2=4x^2$

$a=1 \qquad u=2x$

$\frac{du}{dx}=2 \qquad \frac{1}{2}du=dx$

$\displaystyle\frac{1}{2}\int\frac{du}{a^2+u^2} = \frac{1}{2}\frac{1}{a}\arctan\frac{u}{a}+C$

$\displaystyle = \frac{1}{2}\arctan(2x)+C\Big|_0^{\sqrt{3}/2}$

$= \frac{1}{2}\arctan(\sqrt{3})+C-\left(\frac{1}{2}\arctan 0+C\right)$

$= \frac{1}{2}\cdot\frac{\pi}{3}-\frac{1}{2}\cdot 0 = \boxed{\frac{\pi}{6}}$

8. $\displaystyle\int\frac{1}{x\sqrt{4x^2-1}}\,dx$

$u^2=4x^2 \qquad a^2=1$

$u=2x \qquad a=1$

$\frac{du}{dx}=2 \qquad x=\frac{1}{2}u$

$\frac{1}{2}du=dx$

$\displaystyle\frac{1}{2}\int\frac{du}{\frac{1}{2}u\sqrt{u^2-a^2}} = \int\frac{du}{u\sqrt{u^2-a^2}}$

$\displaystyle = \frac{1}{a}\operatorname{arc\,sec}\frac{|u|}{a}+C$

$= \operatorname{arcsec}\frac{|2x|}{1}+C$

$= \boxed{\operatorname{arcsec}|2x|+C}$

9. $\displaystyle\int\frac{x^3}{x^2+1}\,dx = \int\left(x-\frac{x}{x^2+1}\right)dx = \boxed{\frac{1}{2}x^2-\frac{1}{2}\ln(x^2+1)+C}$

$$\begin{array}{r} x \\ x^2+1\,\overline{)\,x^3} \\ -(x^3+x) \\ \hline -x \end{array}$$

$u=x^2+1$

$\frac{du}{dx}=2x \qquad \frac{1}{2}du=x\,dx$

$\displaystyle\frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|$

10. $\displaystyle\int\frac{1}{\sqrt{1-(x+1)^2}}\,dx$

$a^2=1 \qquad u^2=(x+1)^2$

$a=1 \qquad u=x+1$

$\frac{du}{dx}=1 \qquad du=dx$

$\displaystyle\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C$

$\displaystyle = \arcsin\left(\frac{x+1}{1}\right)+C$

$= \boxed{\arcsin(x+1)+C}$

11. $\displaystyle\int\frac{e^{2x}}{4+e^{4x}}\,dx$

$a^2=4 \qquad u^2=e^{4x}$

$a=2 \qquad u=e^{2x}$

$\frac{du}{dx}=2e^{2x} \qquad \frac{1}{2}du=e^{2x}dx$

$\displaystyle\frac{1}{2}\int\frac{du}{a^2+u^2} = \frac{1}{2}\cdot\frac{1}{a}\arctan\frac{u}{a}+C$

$= \boxed{\frac{1}{4}\arctan\frac{e^{2x}}{2}+C}$

12. $\int \frac{\sin x}{1+\cos^2 x}\,dx$ $\qquad -\int \frac{du}{a^2+u^2} = -\frac{1}{a}\arctan\frac{u}{a}+C$

$a^2=1 \qquad u^2=\cos^2 x$

$a=1 \qquad u=\cos x$

$\frac{du}{dx} = -\sin x \qquad -du = \sin x\,dx$

$= -\frac{1}{1}\arctan\frac{(\cos x)}{1}+C \Big|_{\pi/2}^{\pi}$

$= -\left[\arctan(\cos\pi)+C\right] - \left(\arctan\left(\cos\frac{\pi}{2}\right)+C\right)$

$= -\left[-\pi/4 - 0\right] = \boxed{\frac{\pi}{4}}$

13. $\int_0^2 \frac{dx}{x^2-2x+2}$ $\qquad x^2-2x+1 \quad +2-1 \qquad \left(\frac{-2}{2}\right)^2$

$(x-1)^2 \quad +1$

$\int_0^2 \frac{dx}{(x-1)^2+1}$

$u^2=(x-1)^2 \qquad a^2=1$

$u=x-1 \qquad a=1$

$\frac{du}{dx}=1$

$du=dx$

$\int \frac{du}{u^2+a^2} = \frac{1}{a}\arctan\frac{u}{a}+C$

$= \frac{1}{1}\arctan\frac{x-1}{1}+C\Big|_0^2$

$= \arctan 1 + C - (\arctan(-1)+C)$

$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{2\pi}{4} = \boxed{\frac{\pi}{2}}$

*14. $\int \frac{2x}{x^2+6x+13}\,dx = \int \frac{2x}{(x+3)^2+4}\,dx$

$x^2+6x+9 \quad +13-9$

$(x+3)^2+4$

$\int \frac{2x+6-6}{x^2+6x+13}\,dx = \int \frac{2x+6}{x^2+6x+13}\,dx - \int \frac{6}{(x+3)^2+4}\,dx$

$u=x^2+6x+13 \qquad u^2=(x+3)^2 \quad a^2=4$

$\frac{du}{dx}=2x+6 \qquad u=x+3 \quad a=2$

$du=(2x+6)dx \qquad \frac{du}{dx}=1$

$\int\frac{du}{u}=\ln|u| \qquad du=dx$

$-6\int\frac{du}{u^2+a^2}$

$=-6\cdot\frac{1}{a}\arctan\frac{u}{a}$

$\boxed{\ln|x^2+6x+13| - 3\arctan\frac{x+3}{2}+C}$

*15 $\int \frac{x-2}{(x+1)^2+4}\,dx = \int \frac{x-2}{x^2+2x+5}\,dx$

$x^2+2x+1+4$

$\int \frac{x-2+3-3}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5} - \int \frac{3}{(x+1)^2+4}\,dx$

$u = x^2+2x+5$

$\frac{du}{dx} = 2x+2$

$\frac{1}{2}du = (x+1)dx$

$\frac{1}{2}\int \frac{du}{u}$

$\frac{1}{2}\ln|u|$

$u^2 = (x+1)^2 \quad a^2 = 4$

$u = x+1 \quad a = 2$

$\frac{du}{dx} = 1$

$du = dx$

$-3\int \frac{du}{u^2+a^2}$

$-3\cdot\frac{1}{a}\arctan\frac{u}{a}$

$$\boxed{\frac{1}{2}\ln|x^2+2x+5| - \frac{3}{2}\arctan\frac{x+1}{2} + C}$$