Optimization Problems

1. If the sum of two numbers is 32, find the maximum product of the two numbers.

   256

2. The sum of a number and the cube of a second number is 32. Find the maximum product of the two numbers.

   48

3. A farmer has 400 yards of fencing and wishes to fence three sides of a rectangular field (the fourth side is along an existing stone wall, and needs no additional fencing). Find the dimensions of the rectangular field of largest area that can be fenced.

   max area $= 20000\ yd^2$ 100yd by 200 yd

4. A rectangular field adjacent to a river is to be enclosed. Fencing along the river costs \$5 per meter, and the fencing for the other sides costs \$3 per meter. The area of the field is to be 1200 square meters. Find the dimensions of the field that is the least expensive to enclose.

   40 m by 30 m

   length by river

5. A metal box (without a top) is to be constructed from a square sheet of metal that is 20 cm on a side by cutting square pieces of the same size from the corners of the sheet and then folding up the sides. Find the dimensions of the box with the largest volume that can be constructed in this manner.

   $\frac{10}{3}$ cm by $\frac{40}{3}$ cm by $\frac{40}{3}$ cm

6. A rectangular playing field is to have area 600 m^2. Fencing is required to enclose the field and to divide it into two equal halves.

   a) Find a formula, F(x), for the total length of fencing required, in terms of the length, x, of the fence dividing the field in half. $F(x) = 3x + \frac{1200}{x}$

   b) Find the minimum amount of fencing needed to do this.

   120 m

   c) What are the outer dimensions of the field that has the least fencing?

   20m by 30m

7. Find the closest point on the curve $x^2 + y^2 = 1$ to the point (2, 4).

   $\left(\sqrt{\frac{1}{5}}, \sqrt{\frac{4}{5}}\right)$

8. A rectangle has its base on the x-axis and its upper vertices on the parabola $y = 27 - x^2$. Find the maximum possible area of the rectangle.

   108

9. A rectangular container with open top is required to have a volume of 16 cubic meters. Also, one side of the rectangular base is required to be 4 meters long. If material for the base costs \$8 per square meter, and material for the sides costs \$2 per square meter, find the dimensions of the container so that the cost of material to make it will be a minimum.

   $\sqrt{2}$ m by 4 m by $\frac{4}{\sqrt{2}}$ m

8

10. A rectangular box with open top is to be constructed from a rectangular piece of cardboard 80 cm by 30 cm, by cutting out equal squares from each corner of the sheet of cardboard and folding up the resulting flaps. Find the dimensions of the box of maximum volume made by these conditions.

$\frac{20}{3}$ cm by $\frac{200}{3}$ cm by $\frac{50}{3}$ cm

11. A computer company determines that its profit equation (in millions of dollars) is given by $P = x^3 - 48x^2 + 720x - 1000$, where x is the number of thousands of units of software sold and $0 \le x \le 40$. Optimize the manufacturer's profit.

40 thousands of units of software

12. What is the radius of a cylindrical can with volume 512 cubic inches that will use the minimum material?

$\sqrt[3]{\frac{256}{\pi}}$ in

13. A 4-meter length of stiff wire is cut in two pieces. One piece is bent into the shape of a square and the other into a rectangle whose length is 3 times its width. Let x be the length of the side of the square.

a) Find a formula A(x), the sum of the areas of the square and rectangle, in terms of the variable x.

$A(x) = x^2 + 3\left(\frac{1-x}{2}\right)\left(\frac{1-x}{2}\right)$

b) For what values of x does A(x) achieve its maximum; for which does it achieve its minimum. Justify your answer.

min: $x = \frac{3}{7}$ m

max: $x = 4$m

Free-response Practice: 1972 AB 4/BC 3

A man has 340 yards of fencing for enclosing two separate fields, one of which is to be a rectangle twice as long as it is wide and the other a square. The square field must contain at least 100 square yards and the rectangular one must contain at least 800 square yards.

a. If x is the width of the rectangular field, what are the maximum and minimum possible values of x? min: $x = 20$ yd max: $x = 50$ yd

b. What is the greatest number of square yards that can be enclosed in the two fields? Justify your answer.

See solutions for work !!!

abs. max area = 5100 yd^2

1. Let $x = 1^{st}\#$

$y = 2^{nd}\#$

max. $P = xy$

$x + y = 32$

$y = 32 - x$

$P = x(32 - x) = 32x - x^2$

$P' = 32 - 2x = 0$

$-2x = -32$

$x = 16$

$y = 32 - 16 = 16$

$+ \quad | \quad -$

16

$\boxed{\text{max product} = 256}$

2. Let $x = $ one #

$y = $ other #

max. $P = xy$

$x + y^3 = 32$

$x = 32 - y^3$

$P = (32 - y^3)y$

$P = 32y - y^4$

$P' = 32 - 4y^3 = 0$

$-4y^3 = -32$

$y^3 = 8$

$y = 2$

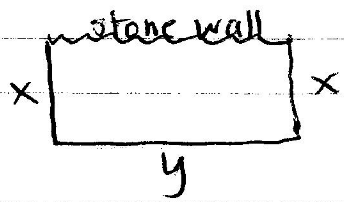
$x = 32 - (2)^3 = 24$

$+ \quad | \quad -$

2

$\boxed{\text{max product} = 48}$

3.



max $A = xy$

$2x + y = 400$

$y = 400 - 2x$

$A = x(400 - 2x) = 400x - 2x^2$

$A' = 400 - 4x = 0$

$-4x = -400$

$x = 100$

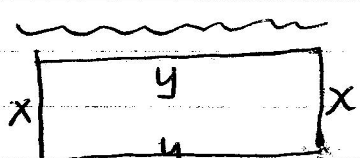
$y = 400 - 2(100) = 200$

$+ \quad | \quad -$

100

$\boxed{\text{max area } 100 \cdot 200 = 20000\,yd^2}$

4.



min. $C = 5y + 3(2x + y)$

$A = xy = 1200$

$y = \frac{1200}{x}$

$C = 5y + 6x + 3y = 8y + 6x$

$C = 8\left(\frac{1200}{x} + 6x\right) = 9600x^{-1} + 6x$

$C' = -9600x^{-2} + 6$

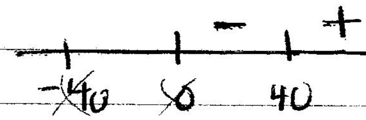
undef. when $x = 0$

$$\frac{-9600}{x^2} + 6 = 0$$

$$-9600 = -6x^2$$

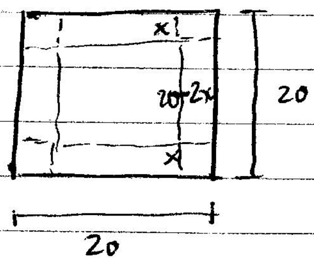
$$x^2 = 1600$$

$$x = \pm 40$$



$$\boxed{\begin{array}{l} x = 40 \text{ m} \\ y = \frac{1200}{40} = 30 \text{ m} \end{array}}$$

5.



$$V = (20-2x)(20-2x)(x) = (400-40x-40x+4x^2)x$$

$$V = 400x - 80x^2 + 4x^3$$

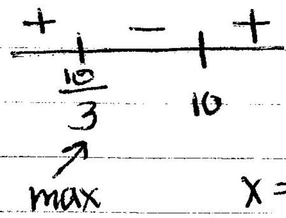
$$V' = 400 - 160x + 12x^2$$

$$4(3x^2 - 40x + 100)$$

$$4(3x-10)(x-10) = 0$$

$$3x - 10 = 0 \quad x - 10 = 0$$

$$x = \frac{10}{3} \quad x = 10$$

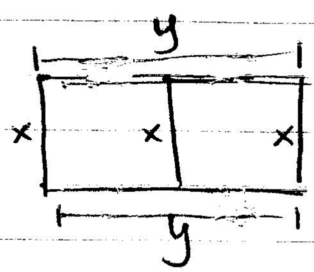


$$x = \frac{10}{3}$$

$$20 - 2x = \frac{40}{3}$$

ht.		length		width
$\frac{10}{3}$ cm	by	$\frac{40}{3}$ cm	by	$\frac{40}{3}$ cm

6.



a) $F(x) = 3x + 2y$ $\qquad xy = 600$

$F(x) = 3x + 2\left(\frac{600}{x}\right)$ $\qquad y = \frac{600}{x}$

$$F(x) = 3x + \frac{1200}{x}$$

b) $F'(x) = 3 - 1200x^{-2}$

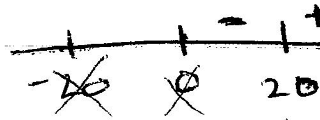
$3 - \frac{1200}{x^2} = 0$ undef when $x = 0$

$$3x^2 - 1200 = 0$$

c) 20m by 30m

$$x^2 = 400$$

$$x = \pm 20$$



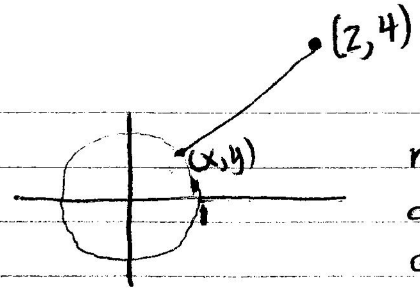
$x = 20$

$y = \frac{600}{20} = 30$

min amt. $= 3(20) + 2(30)$

$= 120$ m

7.



minimize distance

$d = \sqrt{(x-2)^2 + (y-4)^2}$

$d = \sqrt{(x-2)^2 + (\sqrt{1-x^2}-4)^2}$

min. what's under the radical $\Rightarrow$ you min. the distance

$x^2 + y^2 = 1$

$y^2 = 1 - x^2$

$y = \sqrt{1-x^2}$

$(x-2)^2 + (\sqrt{1-x^2}-4)^2$

$\cancel{x^2} - 4x + 4 + 1 - \cancel{x^2} - 8\sqrt{1-x^2} + 16$

$-4x + 21 - 8\sqrt{1-x^2}$

deriv: $-4 - 8 \cdot \frac{1}{2}(1-x^2)^{-1/2}(-2x)$

$-4 + \dfrac{8x}{\sqrt{1-x^2}} = 0$ undef. when $x = 1$ or -1

$-4\sqrt{1-x^2} + 8x = 0$

$(-4\sqrt{1-x^2})^2 = (+8x)^2$

$16(1-x^2) = 64x^2$

$16 = 80x^2$

$x^2 = \dfrac{16}{80} = \dfrac{1}{5}$

$x = \pm\sqrt{\dfrac{1}{5}}$

undef $\quad - \quad - \quad + \quad$ undef

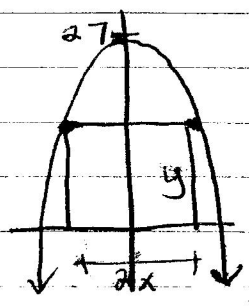
$\cancel{-1} \quad \cancel{-\sqrt{\tfrac{1}{5}}} \quad \sqrt{\tfrac{1}{5}} \quad 1$

min

$x = \sqrt{\frac{1}{5}} \quad y = \sqrt{1-(\sqrt{\frac{1}{5}})^2} = \sqrt{1-\frac{1}{5}} = \sqrt{\frac{4}{5}}$

$\left(\sqrt{\dfrac{1}{5}}, \sqrt{\dfrac{4}{5}}\right)$

8.



$A = 2xy = 2x(27-x^2) = 54x - 2x^3$

$A' = 54 - 6x^2 = 0$

$-6x^2 = -54$

$x^2 = 9$

$x = \pm 3$

$- \quad + \quad -$

$-3 \quad 3$ max

$y = 27 - 3^2 = 18$

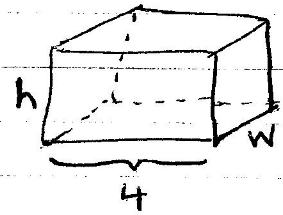
Area $= 2xy = 2(3)(18) = \boxed{108}$

9. $V = 16m^3 = lwh$

let $l = 4m$ $\quad V = 4wh = 16 \Rightarrow h = \frac{4}{w}$

base $= 4w$ $\quad$ sides: $hw * 2 = 2hw$

$4h * 2 = 8h$



$C = 8(4w) + 2(2hw + 8h)$

$C = 32w + 4hw + 16h$

$32w + 4\left(\frac{4}{w}\right)w + 16\left(\frac{4}{w}\right)$

$C = 32w + 16 + \frac{64}{w}$

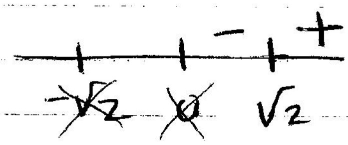
$C' = 32 - 64w^{-2}$ $\quad$ undef. when $w=0$

$32 - \frac{64}{w^2} = 0$

$32w^2 - 64 = 0$

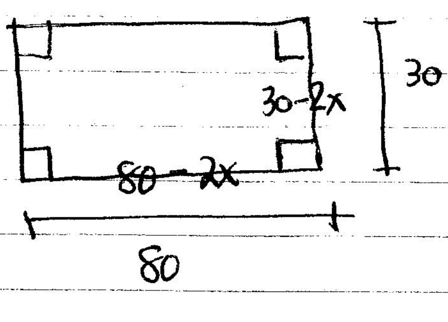
$w^2 = 2$

$w = \pm\sqrt{2}$



$\boxed{\begin{array}{l} w = \sqrt{2}\ m \\ l = 4\ m \\ h = \frac{4}{\sqrt{2}}\ m \end{array}}$

10.



$V = x(30-2x)(80-2x)$

$V = (30x - 2x^2)(80-2x)$

$V' = (30x - 2x^2)(-2) + (80-2x)(30-4x)$

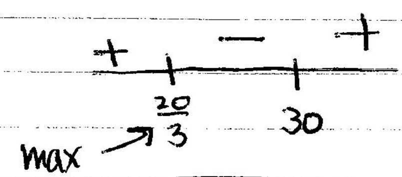
$= -60x + 4x^2 + 2400 - 380x + 8x^2$

$= 12x^2 - 440x + 2400$

$= 4(3x^2 - 110x + 600)$

$= 4(3x - 20)(x - 30)$

$x = \frac{20}{3} \quad x = 30$



$\boxed{\begin{array}{l} ht = \frac{20}{3}\ cm \\ l = 80 - 2x = \frac{200}{3}\ cm \\ w = 30 - 2x = \frac{50}{3}\ cm \end{array}}$

11. $P = x^3 - 48x^2 + 720x - 1000$

$P' = 3x^2 - 96x + 720$

$0 = 3(x^2 - 32x + 240)$

$0 = 3(x-12)(x-20)$

$x = 12 \quad x = 20$

x	$P(x)$
0	-1000
12	2456
40	15000

$[\ + \ \big|_{12}\ - \ \big|_{20}\ + \ ]$ on $[0, 40]$

rel max → 12

$\boxed{\text{max profit when 40 thousands of units are sold}}$

12. $V = \pi r^2 h = 512 \Rightarrow h = \frac{512}{\pi r^2}$

minimize surface area

$S = 2\pi r^2 + 2\pi r h$

$S = 2\pi r^2 + 2\pi r\left(\frac{512}{\pi r^2}\right) = 2\pi r^2 + \frac{1024}{r}$

$S' = 2\pi \cdot 2r + 1024 \cdot -1r^{-2}$

$= 4\pi r \quad \frac{1024}{r^2}$ undef. if $r = 0$

$4\pi r - \frac{1024}{r^2} = 0$

$4\pi r^3 - 1024 = 0$

$4\pi r^3 = 1024$

$r^3 = \frac{256}{\pi}$

$r = \sqrt[3]{\frac{256}{\pi}}$

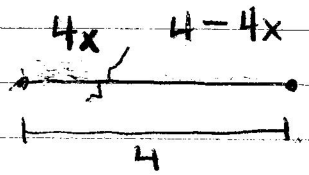
$\cancel{0}\ \big|\ - \ \big|_{\sqrt[3]{256/\pi}}\ +$

$\sqrt[3]{\frac{256}{\pi}}$ ← min

4.335

$\boxed{r = \sqrt[3]{\frac{256}{\pi}} \text{ in}}$

13.



side sq $= x \Rightarrow$ perim $= 4x$

leaves $4 - 4x$ to form rectangle

rect: $l = 3w \quad$ perim $= 2l + 2w = 2(3w) + 2w = 8w$

$8w = 4 - 4x \quad w = \frac{4-4x}{8} = \frac{1-x}{2}$

a) $A(x) = x^2 + 3\left(\frac{1-x}{2}\right)\left(\frac{1-x}{2}\right)$

$= x^2 + \frac{3}{4}(1 - 2x + x^2)$

b) $A'(x) = 2x + \frac{3}{4}(-2 + 2x)$

$2x - \frac{3}{2} + \frac{3}{2}x = 0$

$\frac{7}{2}x = \frac{3}{2}$

$x = \frac{3}{7}$

$-\ \big|_{3/7}\ +$

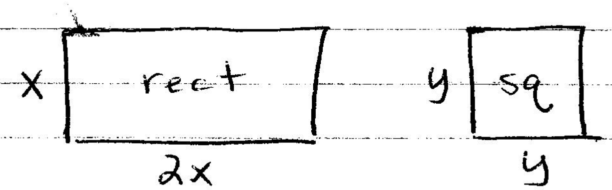
$3/7$ ← min

$A(x)$ is a min when $x = \frac{3}{7}$ m

$A(x)$ is a max when $x = 4$ m (so there would only be a square & no rectangle)

x	$A(x)$	
0	3	
3/7	1.7143	$\leftarrow$ abs. min
4	91	$\leftarrow$ abs. max

Free Response 1972 AB4/BC3



a) perim $= 6x + 4y = 340 \Rightarrow 4y = 340 - 6x$

$y = 85 - \frac{3}{2}x$

$y^2 \geq 100 \Rightarrow y \geq 10$

$2x^2 \geq 800 \Rightarrow x \geq 20$

$85 - \frac{3}{2}x \geq 10$

$-\frac{3}{2}x \geq -75$

$x \leq 50$

$$\boxed{\begin{array}{l} \text{min. } x = 20 \text{ yd} \\ \text{max } x = 50 \text{ yd} \end{array}}$$

b) maximize the total area

$$A = 2x^2 + y^2 = 2x^2 + \left(85 - \frac{3}{2}x\right)^2$$

$$A' = 4x + 2\left(85 - \frac{3}{2}x\right)^1\left(-\frac{3}{2}\right) = 4x - 255 + \frac{9}{2}x = 0$$

$\frac{17}{2}x = 255$

$x = 30$

$-$ | $+$

30 $\leftarrow$ min

abs. max must occur at an endpt since $x = 30$ gives a min

x	$A(x)$
20	3825
50	5100

$\boxed{5100 \text{ yd}^2}$