76. Let h be the length of the altitude to base b and denote the area of the triangle by A. Then

$$\frac{h}{a} = \sin\theta$$

$$\therefore h = a\sin\theta$$

Since $A = \frac{1}{2}bh$, we can substitute $h = a\sin\theta$ to get

$A = \frac{1}{2}ab\sin\theta$.

Section 4.3 Trigonometry Extended: The Circular Functions

Exploration 1

1. The side opposite θ in the triangle has length y and the hypotenuse has length r. Therefore

$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r}$.

2. $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$

3. $\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{y}{x}$

4. $\cot\theta = \frac{x}{y}$; $\sec\theta = \frac{r}{x}$; $\csc\theta = \frac{r}{y}$

Exploration 2

1. The x-coordinates on the unit circle lie between -1 and 1, and $\cos t$ is always an x-coordinate on the unit circle.

2. The y-coordinates on the unit circle lie between -1 and 1, and $\sin t$ is always a y-coordinate on the unit circle.

3. The points corresponding to t and $-t$ on the number line are wrapped to points above and below the x-axis with the same x-coordinates. Therefore $\cos t$ and $\cos(-t)$ are equal.

4. The points corresponding to t and $-t$ on the number line are wrapped to points above and below the x-axis with exactly opposite y-coordinates. Therefore $\sin t$ and $\sin(-t)$ are opposites.

5. Since 2π is the distance around the unit circle, both t and $t + 2\pi$ get wrapped to the same point.

6. The points corresponding to t and $t + \pi$ get wrapped to points on either end of a diameter on the unit circle. These points are symmetric with respect to the origin and therefore have coordinates (x, y) and $(-x, -y)$. Therefore $\sin t$ and $\sin(t + \pi)$ are opposites, as are $\cos t$ and $\cos(t + \pi)$.

7. By the observation in (6), $\tan t$ and $\tan(t + \pi)$ are ratios of the form $\frac{y}{x}$ and $\frac{-y}{-x}$, which are either equal to each other or both undefined.

8. The sum is always of the form $x^2 + y^2$ for some (x, y) on the unit circle. Since the equation of the unit circle is $x^2 + y^2 = 1$, the sum is always 1.

9. Answers will vary. For example, there are similar statements that can be made about the functions cot, sec, and csc.

Quick Review 4.3

1. Quadrant IV

2. Quadrant III

3. Quadrant I $\left(\text{coterminal with } \frac{\pi}{4}\right)$

4. Quadrant III $\left(\text{coterminal with } \frac{4\pi}{3}\right)$

5. $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

6. $\cot\frac{\pi}{4} = 1$

7. $\csc\frac{\pi}{4} = \sqrt{2}$

8. $\sec\frac{\pi}{3} = 2$

9. Using a right triangle with hypotenuse 13 and legs 5 (opposite) and $\sqrt{13^2 - 5^2} = 12$ (adjacent), we have $\sin\theta = \frac{5}{13}$, $\cos\theta = \frac{12}{13}$, $\tan\theta = \frac{5}{12}$; $\csc\theta = \frac{13}{5}$, $\sec\theta = \frac{13}{12}$, $\cot\theta = \frac{12}{5}$.

10. Using a right triangle with hypotenuse 17 and legs 15 (adjacent) and $\sqrt{17^2 - 15^2} = 8$ (opposite), we have $\sin\theta = \frac{8}{17}$, $\cos\theta = \frac{15}{17}$, $\tan\theta = \frac{8}{15}$; $\csc\theta = \frac{17}{8}$, $\sec\theta = \frac{17}{15}$, $\cot\theta = \frac{15}{8}$.

Section 4.3 Exercises

1. The 450° angle lies on the positive-y axis ($450° - 360° = 90°$), while the others are all coterminal in Quadrant II.

2. The $-\frac{5\pi}{3}$ angle lies in Quadrant I $\left(-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3}\right)$, while the others are all coterminal in Quadrant IV.

In #3–12, recall that the distance from the origin is $r = \sqrt{x^2 + y^2}$.

3. $\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = -\frac{1}{\sqrt{5}}$, $\tan\theta = -2$; $\csc\theta = \frac{\sqrt{5}}{2}$, $\sec\theta = -\sqrt{5}$, $\cot\theta = -\frac{1}{2}$.

4. $\sin\theta = -\frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = -\frac{3}{4}$; $\csc\theta = -\frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = -\frac{4}{3}$.

5. $\sin\theta = -\frac{1}{\sqrt{2}}$, $\cos\theta = -\frac{1}{\sqrt{2}}$, $\tan\theta = 1$; $\csc\theta = -\sqrt{2}$, $\sec\theta = -\sqrt{2}$, $\cot\theta = 1$.

6. $\sin\theta = -\frac{5}{\sqrt{34}}$, $\cos\theta = \frac{3}{\sqrt{34}}$, $\tan\theta = -\frac{5}{3}$; $\csc\theta = -\frac{\sqrt{34}}{5}$, $\sec\theta = \frac{\sqrt{34}}{3}$, $\cot\theta = -\frac{3}{5}$.

7. $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}; \csc\theta = \frac{5}{4}, \sec\theta = \frac{5}{3}, \cot\theta = \frac{3}{4}$.

8. $\sin\theta = -\frac{3}{\sqrt{13}}, \cos\theta = -\frac{2}{\sqrt{13}}, \tan\theta = \frac{3}{2}; \csc\theta = -\frac{\sqrt{13}}{3}, \sec\theta = -\frac{\sqrt{13}}{2}, \cot\theta = \frac{2}{3}$.

9. $\sin\theta = 1, \cos\theta = 0, \tan\theta$ undefined; $\csc\theta = 1$, $\sec\theta$ undefined, $\cot\theta = 0$.

10. $\sin\theta = 0, \cos\theta = -1, \tan\theta = 0$; $\csc\theta$ undefined, $\sec\theta = -1$, $\cot\theta$ undefined.

11. $\sin\theta = -\frac{2}{\sqrt{29}}, \cos\theta = \frac{5}{\sqrt{29}}, \tan\theta = -\frac{2}{5}; \csc\theta = -\frac{\sqrt{29}}{2}, \sec\theta = \frac{\sqrt{29}}{5}, \cot\theta = -\frac{5}{2}$.

12. $\sin\theta = -\frac{1}{\sqrt{2}}, \cos\theta = \frac{1}{\sqrt{2}}, \tan\theta = -1; \csc\theta = -\sqrt{2}, \sec\theta = \sqrt{2}, \cot\theta = -1$.

For #13–16, determine the quadrant(s) of angles with the given measures, and then use the fact that $\sin t$ is positive when the terminal side of the angle is above the x-axis (in quadrants I and II) and $\cos t$ is positive when the terminal side of the angle is to the right of the y-axis (in quadrants I and IV). Note that since $\tan t = \frac{\sin t}{\cos t}$, the sign of $\tan t$ can be determined from the signs of $\sin t$ and $\cos t$: if $\sin t$ and $\cos t$ have the same sign, the answer to (c) will be '+'; otherwise it will be '−'. Thus $\tan t$ is positive in quadrants I and III.

13. These angles are in quadrant I. **(a)** + (i.e., $\sin t > 0$). **(b)** + (i.e., $\cos t > 0$). **(c)** + (i.e., $\tan t > 0$).

14. These angles are in quadrant II. **(a)** +. **(b)** −. **(c)** −.

15. These angles are in quadrant III. **(a)** −. **(b)** −. **(c)** +.

16. These angles are in quadrant IV. **(a)** −. **(b)** +. **(c)** −.

For #17–20, use strategies similar to those for the previous problem set.

17. 143° is in quadrant II, so $\cos 143° < 0$.

18. 192° is in quadrant III, so $\tan 192° > 0$.

19. $\frac{7\pi}{8}$ rad is in quadrant II, so $\cos\frac{7\pi}{8} < 0$.

20. $\frac{4\pi}{5}$ rad is in quadrant II, so $\tan\frac{4\pi}{5} < 0$.

21. (a) $(2, 2)$

22. (b) $(-1, \sqrt{3})$

23. (a) $(-\sqrt{3}, -1)$

24. (b) $(1, -\sqrt{3})$

For #25–36, recall that the reference angle is the acute angle formed by the terminal side of the angle in standard position and the x-axis.

25. The reference angle is 60°. A right triangle with a 60° angle at the origin has the point $P(-1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\cos 120° = \frac{x}{r} = -\frac{1}{2}$.

26. The reference angle is 60°. A right triangle with a 60° angle at the origin has the point $P(1, -\sqrt{3})$ as one vertex, so $\tan 300° = \frac{y}{x} = -\sqrt{3}$.

27. The reference angle is the given angle, $\frac{\pi}{3}$. A right triangle with a $\frac{\pi}{3}$ radian angle at the origin has the point $P(1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\sec\frac{\pi}{3} = \frac{r}{x} = 2$.

28. The reference angle is $\frac{\pi}{4}$. A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(1, 1)$ as one vertex, with hypotenuse length $r = \sqrt{2}$, so $\csc\frac{3\pi}{4} = \frac{r}{y} = \sqrt{2}$.

29. The reference angle is $\frac{\pi}{6}$ (in fact, the given angle is coterminal with $\frac{\pi}{6}$). A right triangle with a $\frac{\pi}{6}$ radian angle at the origin has the point $P(\sqrt{3}, 1)$ as one vertex, with hypotenuse length $r = 2$, so $\sin\frac{13\pi}{6} = \frac{y}{r} = \frac{1}{2}$.

30. The reference angle is $\frac{\pi}{3}$ (in fact, the given angle is coterminal with $\frac{\pi}{3}$). A right triangle with a $\frac{\pi}{3}$ radian angle at the origin has the point $P(1, \sqrt{3})$ as one vertex, with hypotenuse length $r = 2$, so $\cos\frac{7\pi}{3} = \frac{x}{r} = \frac{1}{2}$.

31. The reference angle is $\frac{\pi}{4}$ (in fact, the given angle is coterminal with $\frac{\pi}{4}$). A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(1, 1)$ as one vertex, so $\tan\frac{-15\pi}{4} = \frac{y}{x} = 1$.

32. The reference angle is $\frac{\pi}{4}$. A right triangle with a $\frac{\pi}{4}$ radian angle at the origin has the point $P(-1, -1)$ as one vertex, so $\cot\frac{13\pi}{4} = \frac{x}{y} = 1$.

33. $\cos\frac{23\pi}{6} = \cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$

34. $\cos\frac{17\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

35. $\sin\frac{11\pi}{3} = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$

36. $\cot\frac{19\pi}{6} = \cot\frac{7\pi}{6} = \sqrt{3}$

37. −450° is coterminal with 270°, on the negative y-axis. **(a)** −1. **(b)** 0. **(c)** undefined.

38. −270° is coterminal with 90°, on the positive y-axis. **(a)** 1. **(b)** 0. **(c)** undefined.

39. 7π radians is coterminal with π radians, on the negative x-axis. **(a)** 0. **(b)** −1. **(c)** 0.

40. $\frac{11\pi}{2}$ radians is coterminal with $\frac{3\pi}{2}$ radians on the negative y-axis. **(a)** -1. **(b)** 0. **(c)** undefined.

41. $\frac{-7\pi}{2}$ radians is conterminal with $\frac{\pi}{2}$ radians on the positive y-axis. **(a)** 1. **(b)** 0. **(c)** undefined.

42. -4π radians is conterminal with 0 radians, on the positive x-axis. **(a)** 0. **(b)** 1. **(c)** 0.

43. Since $\cot\theta > 0$, $\sin\theta$ and $\cos\theta$ have the same sign, so $\sin\theta = +\sqrt{1-\cos^2\theta} = \frac{\sqrt{5}}{3}$, and $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{5}}{2}$.

44. Since $\tan\theta < 0$, $\sin\theta$ and $\cos\theta$ have opposite signs, so $\cos\theta = -\sqrt{1-\sin^2\theta} = -\frac{\sqrt{15}}{4}$, and $\cot\theta = \frac{\cos\theta}{\sin\theta} = -\sqrt{15}$.

45. $\cos\theta = +\sqrt{1-\sin^2\theta} = \frac{\sqrt{21}}{5}$, so $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{2}{\sqrt{21}}$ and $\sec\theta = \frac{1}{\cos\theta} = \frac{5}{\sqrt{21}}$.

46. $\sec\theta$ has the same sign as $\cos\theta$, and since $\cot\theta > 0$, $\sin\theta$ must also be negative. With $x = -3$, $y = -7$, and $r = \sqrt{3^2+7^2} = \sqrt{58}$, we have $\sin\theta = -\frac{7}{\sqrt{58}}$ and $\cos\theta = -\frac{3}{\sqrt{58}}$.

47. Since $\cos\theta < 0$ and $\cot\theta < 0$, $\sin\theta$ must be positive. With $x = -4$, $y = 3$, and $r = \sqrt{4^2+3^2} = 5$, we have $\sec\theta = -\frac{5}{4}$ and $\csc\theta = \frac{5}{3}$.

48. Since $\sin\theta > 0$ and $\tan\theta < 0$, $\cos\theta$ must be negative. With $x = -3$, $y = 4$, and $r = \sqrt{4^2+3^2} = 5$, we have $\csc\theta = \frac{5}{4}$ and $\cot\theta = -\frac{3}{4}$.

49. $\sin\left(\frac{\pi}{6} + 49000\pi\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

50. $\tan(1234567\pi) - \tan(7654321\pi) = \tan(\pi) - \tan(\pi) = 0$

51. $\cos\left(\frac{5555555\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$

52. $\tan\left(\frac{3\pi - 70000\pi}{2}\right) = \tan\left(\frac{3\pi}{2}\right) =$ undefined.

53. The calculator's value of the irrational number π is necessarily an approximation. When multiplied by a very large number, the slight error of the original approximation is magnified sufficiently to throw the trigonometric functions off.

54. $\sin t$ is the y-coordinate of the point on the unit circle after measuring counterclockwise t units from $(1, 0)$. This will repeat every 2π units (and not before), since the distance around the circle is 2π.

55. $\mu = \frac{\sin 83^\circ}{\sin 36^\circ} \approx 1.69$

56. $\sin\theta_2 = \frac{\sin 42^\circ}{1.52} \approx 0.44$, so $\theta_2 \approx 26.12^\circ$.

57. **(a)** When $t = 0$, $d = 0.4$ in.

(b) When $t = 3$, $d = 0.4e^{-0.6}\cos 12 \approx 0.1852$ in.

58. When $t = 0$, $\theta = 0.25$ (rad). When $t = 2.5$, $\theta = 0.25\cos 2.5 \approx -0.2003$ rad.

59. The difference in the elevations is 600 ft, so $d = 600/\sin\theta$. Then:

(a) $d = 600\sqrt{2} \approx 848.53$ ft.

(b) $d = 600$ ft.

(c) d ≈ 933.43 ft.

60. January ($t = 1$): $72.4 + 61.7\sin\frac{\pi}{6} = 103.25$.

April ($t = 4$): $72.4 + 61.7\sin\frac{2\pi}{3} \approx 125.83$.

June ($t = 6$): $72.4 + 61.7\sin\pi = 72.4$.

October ($t = 10$): $72.4 + 61.7\sin\frac{5\pi}{3} \approx 18.97$.

December ($t = 12$): $72.4 + 61.7\sin 2\pi = 72.4$. June and December are the same; perhaps by June most people have suits for the summer, and by December they are beginning to purchase them for next summer (or as Christmas presents, or for mid-winter vacations).

61. True. Any angle in a triangle measures between 0° and 180°. Acute angles (<90°) determine reference triangles in Quadrant I, where the cosine is positive, while obtuse angles (>90°) determine reference triangles in Quadrant II, where the cosine is negative.

62. True. The point determines a reference triangle in Quadrant IV, with $r = \sqrt{8^2 + (-6)^2} = 10$. Thus $\sin\theta = y/x = -6/10 = -0.6$

63. If $\sin\theta = 0.4$, then $\sin(-\theta) + \csc\theta = -\sin\theta + \frac{1}{\sin\theta} = -0.4 + \frac{1}{0.4} = 2.1$. The answer is (e).

64. If $\cos\theta = 0.4$, then $\cos(\theta + \pi) = -\cos\theta = -0.4$. The answer is (b).

65. $(\sin t)^2 + (\cos t)^2 = 1$ for all t. The answer is (a).

66. $\sin\theta = -\sqrt{1-\cos^2\theta}$, because $\tan\theta = (\sin\theta)/(\cos\theta) > 0$. So $\sin\theta = -\sqrt{1 - \frac{25}{169}} = -\frac{12}{13}$. The answer is (a).

67. Since $\sin\theta > 0$ and $\tan\theta < 0$, the terminal side must be in quadrant II, so $\theta = \frac{5\pi}{6}$.

68. Since $\cos\theta > 0$ and $\sin\theta < 0$, the terminal side must be in quadrant IV, so $\theta = \frac{11\pi}{6}$.