

Related Rates & Implicit Differentiation Free Response Questions

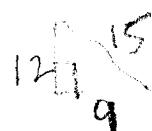
1982 AB4
Solution

(a) $x^2 + y^2 = 15^2$

Implicit: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$

$$9 \cdot \frac{1}{2} + 12 \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{3}{8}$$



(b) $A = \frac{1}{2}xy$

Implicit: $\frac{dA}{dt} = \frac{1}{2} \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right)$

$$\frac{dA}{dt} = \frac{1}{2} \left(9 \cdot \left(-\frac{3}{8} \right) + 12 \cdot \frac{1}{2} \right)$$

$$\frac{dA}{dt} = \frac{21}{16}$$

1978 AB5/BC1

Solution

(a) Implicit differentiation gives

$$2x - xy' - y + 2yy' = 0$$

$$(2y - x)y' = y - 2x$$

$$\boxed{y' = \frac{y - 2x}{2y - x}}$$

$$x^2 - xy + y^2 = 9$$

$$(2y)^2 - (2y)y + y^2 = 9$$

$$3y^2 = 9 \quad y^2 = 3 \quad y = \pm \sqrt{3}$$

(b) There is a vertical tangent when $2y - x = 0$, so $x = 2y$. Substituting into the equation of the curve gives $(2y)^2 - (2y)y + y^2 = 9$, or $3y^2 = 9$. Therefore $y = \pm \sqrt{3}$ and the two points on the curve where the tangents are vertical are $(2\sqrt{3}, \sqrt{3})$ and $(-2\sqrt{3}, -\sqrt{3})$.

$$(c) \quad y'' = \frac{(2y - x)(y' - 2) - (y - 2x)(2y' - 1)}{(2y - x)^2}$$

$$\text{At the point } (0, 3), \quad y' = \frac{3-0}{6-0} = \frac{1}{2} \quad \text{and so} \quad y'' = \frac{(6-0)\left(\frac{1}{2}-2\right) - (3-0)(1-1)}{(6-0)^2} = \boxed{-\frac{1}{4}}$$

Alternatively, one can use implicit differentiation a second time to get

$$2 - xy'' - y' - y' + 2yy' + 2(y')^2 = 0$$

Substituting $x = 0$, $y = 3$, and $y' = \frac{1}{2}$ gives

$$2 - 0 - \frac{1}{2} - \frac{1}{2} + 6y'' + 2\left(\frac{1}{4}\right) = 0 \Rightarrow 6y'' = -\frac{3}{2} \Rightarrow y'' = -\frac{1}{4}$$