

DEFINITION --- Dot product

The **dot product** or **inner product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

Ex 1) Find each dot product:

a) $\langle 3, 4 \rangle \cdot \langle 5, 2 \rangle$

$$\begin{array}{r} 3 \cdot 5 + 4 \cdot 2 \\ 15 + 8 \\ \hline 23 \end{array}$$

b) $\langle 1, -2 \rangle \cdot \langle -4, 3 \rangle$

$$\begin{array}{r} 1 \cdot -4 + -2 \cdot 3 \\ -4 + -6 \\ \hline -10 \end{array}$$

c) $\langle 2\mathbf{i} - \mathbf{j} \rangle \cdot \langle 3\mathbf{i} - 5\mathbf{j} \rangle$

$$\begin{array}{r} \langle 2, -1 \rangle \cdot \langle 3, -5 \rangle \\ 2 \cdot 3 + -1 \cdot -5 \\ 6 + 5 \\ \hline 11 \end{array}$$

Properties of the Dot product ----- Let \mathbf{u} , \mathbf{v} , and \mathbf{w} be vectors and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

4. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

2. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$

($\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

3. $\mathbf{0} \cdot \mathbf{u} = 0$

5. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (cv) = c(\mathbf{u} \cdot \mathbf{v})$

Ex 2) Use the dot product to find the length of vector $\mathbf{u} = \langle 4, -3 \rangle$

$$\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}} = |\mathbf{u}|^2 \text{ then } |\mathbf{u}| = \sqrt{\overrightarrow{\mathbf{u}} \cdot \overrightarrow{\mathbf{u}}}$$

$$\begin{aligned} \sqrt{\langle 4, -3 \rangle \cdot \langle 4, -3 \rangle} &= \sqrt{4 \cdot 4 + -3 \cdot -3} \\ &= \sqrt{16 + 9} = \sqrt{25} = 5 \end{aligned}$$

Angle Between Two Vectors

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \quad \text{and} \quad \theta = \cos^{-1} \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} \right)$$

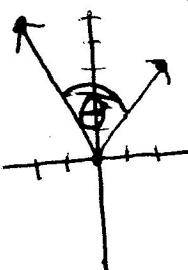
Ex 3) Find the angle between two vectors \mathbf{u} & \mathbf{v} .

a) $\mathbf{u} = \langle 2, 3 \rangle, \mathbf{v} = \langle -2, 5 \rangle$

b) $\mathbf{u} = \langle 2, 1 \rangle, \mathbf{v} = \langle -1, -3 \rangle$

$$\theta = \cos^{-1} \left(\frac{\langle 2, 3 \rangle \cdot \langle -2, 5 \rangle}{\sqrt{2^2 + 3^2} \cdot \sqrt{(-2)^2 + 5^2}} \right)$$

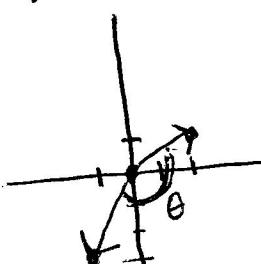
$$\begin{aligned} \theta &= \cos^{-1} \left(\frac{\langle 2, 1 \rangle \cdot \langle -1, -3 \rangle}{\sqrt{2^2 + 1^2} \cdot \sqrt{(-1)^2 + (-3)^2}} \right) \\ &= \cos^{-1} \left(\frac{-2 + -3}{\sqrt{5} \cdot \sqrt{10}} \right) \end{aligned}$$



$$\begin{aligned} &= \cos^{-1} \left(\frac{-4 + 15}{\sqrt{13} \cdot \sqrt{29}} \right) \\ &= \cos^{-1} \left(\frac{11}{\sqrt{377}} \right) \end{aligned}$$

$$= \cos^{-1} \left(\frac{11}{\sqrt{377}} \right)$$

$$= 55.5^\circ$$

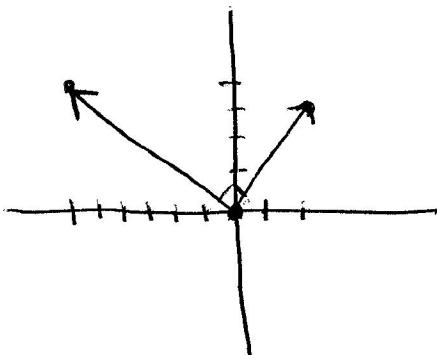


$$= \cos^{-1} \left(\frac{-5}{\sqrt{50}} \right)$$

$$135^\circ$$

right angle
Definition ----- Orthogonal Vectors → The vectors \mathbf{u} and \mathbf{v} are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Ex 4) Prove that the vectors $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle -6, 4 \rangle$ are orthogonal.



$$\langle 2, 3 \rangle \cdot \langle -6, 4 \rangle$$

$$-12 + 12 \\ 0$$

Projection of \mathbf{u} onto \mathbf{v} ----- If \mathbf{u} and \mathbf{v} are nonzero vectors, the projection of \mathbf{u} onto \mathbf{v} is $\text{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$

Ex 5) Find the vector projection of $\mathbf{u} = \langle 6, 2 \rangle$ onto $\mathbf{v} = \langle 5, -5 \rangle$. Then write \mathbf{u} as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}} \mathbf{u}$.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{30 + -10}{(\sqrt{5^2 + (-5)^2})^2} \cdot \langle 5, -5 \rangle$$

$$= \frac{20}{(\sqrt{50})^2} \cdot \langle 5, -5 \rangle$$

$$= \frac{2}{5} \langle 5, -5 \rangle$$

$$\boxed{\langle 2, -2 \rangle}$$

$$\vec{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + \vec{w}$$

$$\langle 6, 2 \rangle = \langle 2, -2 \rangle + \vec{w}$$

$$\vec{w} = \langle 6, 2 \rangle - \langle 2, -2 \rangle$$

$$\vec{w} = \langle 4, 4 \rangle$$

$$\langle 6, 2 \rangle = \langle 2, -2 \rangle + \langle 4, 4 \rangle$$

Ex 6) Find the vector projection of $\mathbf{u} = \langle -3, 4 \rangle$ onto $\mathbf{v} = \langle 12, -5 \rangle$. Then write \mathbf{u} as the sum of two orthogonal vectors, one of which is $\text{proj}_{\mathbf{v}} \mathbf{u}$.

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{-36 + -20}{(\sqrt{12^2 + (-5)^2})^2} \cdot \langle 12, -5 \rangle$$

$$= \frac{-56}{(\sqrt{169})^2} \cdot \langle 12, -5 \rangle$$

$$= \frac{-56}{169} \cdot \langle 12, -5 \rangle$$

$$= \left\langle -\frac{672}{169}, +\frac{280}{169} \right\rangle$$

$$\vec{u} = \text{proj}_{\mathbf{v}} \mathbf{u} + \vec{w}$$

$$\langle -3, 4 \rangle = \left\langle -\frac{672}{169}, \frac{280}{169} \right\rangle + \vec{w}$$

$$\vec{w} = \left\langle \frac{165}{169}, \frac{396}{169} \right\rangle$$

$$\langle -3, 4 \rangle = \left\langle -\frac{672}{169}, \frac{280}{169} \right\rangle + \left\langle \frac{165}{169}, \frac{396}{169} \right\rangle$$