Notes (Section 5.4) - Double & Half Angle Formulas

Double-Angle Identities

$$\sin 2u = 2 \sin u \cos u$$

$$\cos 2u = \begin{cases} \cos^2 u - \sin^2 u \\ 2\cos^2 u - 1 \\ 1 - 2\sin^2 u \end{cases}$$

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}$$

Power-Reducing Identities

$$\sin^2 u = \frac{1 - \cos 2u}{2}$$

$$\cos^2 u = \frac{1 + \cos 2u}{2}$$

$$\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$$

Half-Angle Identities

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}}$$

$$\cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \begin{cases} \pm\sqrt{\dfrac{1 - \cos u}{1 + \cos u}} \\ \dfrac{1 - \cos u}{\sin u} \\ \dfrac{\sin u}{1 + \cos u} \end{cases}$$

depends on quadrant that $\frac{u}{2}$ is in

Let's prove some of these identities with sum and difference formulas.

1) Prove the identity: $\sin 2u = 2 \sin u \cos u$.

$\sin 2u$
$\sin(u+u)$
$\sin u\cos u + \cos u \sin u$
$2\sin u\cos u$

2) Prove the identity: $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

$\cos 2\theta$
$\cos(\theta+\theta)$
$\cos\theta\cdot\cos\theta - \sin\theta\cdot\sin\theta$
$\cos^2\theta - \sin^2\theta$

Now, let's use the new identities to solve the following:

3) Solve $\sin^2 x = 2\sin^2\left(\frac{x}{2}\right)$. over $[0, 2\pi)$

4) Solve algebraically in the interval $[0,2\pi)$: $\sin 2x = \cos x$.

Use half-angle identities to find the exact value of each function:

5) $\cos\frac{\pi}{8}$

6) $\tan 15°$

7) $\sin 75°$

8) $\cos\frac{11\pi}{12}$

Use double-angle identities to find the exact value of each function:

9) $\cos\frac{4\pi}{3}$

10) $\tan 60°$

11) $\sin 120°$

12) ~~$\sin\frac{5\pi}{3}$~~

Use double-angle identities to find the exact values of the function:

13) If $\sin\theta = \frac{3}{5}$ and θ has its terminal side in the first quadrant, find the exact value of each function.

a. $\sin 2\theta$

b. $\cos 2\theta$

c. $\tan 2\theta$

14) If $\tan\theta = \frac{4}{3}$ and $\pi \le \theta \le \frac{3\pi}{2}$ find the exact value of each function.

a. $\sin\left(\frac{\theta}{2}\right)$

b. $\cos\left(\frac{\theta}{2}\right)$

c. $\tan\left(\frac{\theta}{2}\right)$

3) $\sin^2 x = 2\sin^2\left(\frac{x}{2}\right)$

$\sin^2 x = 2\left(\pm\sqrt{\frac{1-\cos x}{2}}\right)^2$

$\sin^2 x = 2 \cdot \frac{1-\cos x}{2}$

$\sin^2 x = 1 - \cos x$

$1 - \cos^2 x = 1 - \cos x$

$0 = \cos^2 x - \cos x$

$0 = \cos x\,(\cos x - 1)$

$\cos x = 0 \qquad \cos x - 1 = 0$

$x = \cos^{-1}(0) \qquad x = \cos^{-1}(1)$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = 0$

4) $\sin 2x = \cos x$

$2\sin x\cos x = \cos x$

$2\sin x\cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0 \qquad 2\sin x - 1 = 0$

$x = \cos^{-1}(0) \qquad x = \sin^{-1}\left(\frac{1}{2}\right)$

$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{\pi}{6}, \frac{5\pi}{6}$

5) $\cos\left(\frac{\pi}{8}\right)$

$\cos(22.5^\circ)$ → 1st quad pos

$\cos\left(\frac{45^\circ}{2}\right)$

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$+\sqrt{\frac{1+\cos 45^\circ}{2}}$

$\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}$

$\sqrt{\frac{\frac{2+\sqrt{2}}{2}}{\frac{2}{1}}}$

$\sqrt{\frac{2+\sqrt{2}}{2} \cdot \frac{1}{2}} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{\sqrt{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

6) $\tan 15^\circ$

$\tan\left(\frac{30^\circ}{2}\right)$

$\frac{1-\cos 30^\circ}{\sin 30^\circ}$

$\frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}}$

$\frac{2-\sqrt{3}}{2} \cdot \frac{2}{1} = 2-\sqrt{3}$

7) $\sin 75^\circ$ 1st quad pos

$\sin\left(\frac{150^\circ}{2}\right)$

$+\sqrt{\frac{1-\cos 150^\circ}{2}}$

$\sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}}$

$\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$

$\sqrt{\frac{2+\sqrt{3}}{2}\cdot\frac{1}{2}} = \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

8) $\cos\left(\frac{11\pi}{12}\right)$

$\cos(165^\circ)$ 2nd quad neg

$\cos\left(\frac{330^\circ}{2}\right)$

$-\sqrt{\frac{1+\cos 330^\circ}{2}}$

$-\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}$

$-\sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$

$-\sqrt{\frac{2+\sqrt{3}}{4}} = \frac{-\sqrt{2+\sqrt{3}}}{2}$

9) $\cos\left(\frac{4\pi}{3}\right)$

$\cos(240^\circ)$

$\cos(2\cdot 120^\circ)$

$2\cos^2 120^\circ - 1$

$2\left(-\frac{1}{2}\right)^2 - 1$

$2\cdot\frac{1}{4} - 1$

$\frac{1}{2} - 1$

$-\frac{1}{2}$

10) $\tan 60^\circ$

$\tan(2\cdot 30^\circ)$

$\frac{2\tan 30^\circ}{1-(\tan 30^\circ)^2}$

$\frac{2\cdot\frac{\sqrt{3}}{3}}{1-\left(\frac{\sqrt{3}}{3}\right)^2}$

$\frac{\frac{2\sqrt{3}}{3}}{1-\frac{3}{9}}$

$\frac{\frac{2\sqrt{3}}{3}}{\frac{2}{3}} = \frac{2\sqrt{3}}{3}\cdot\frac{3}{2} = \sqrt{3}$

11) $\sin 120^\circ$

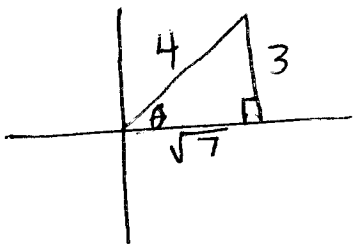
$\sin(2\cdot 60^\circ)$

$2\sin 60^\circ\cos 60^\circ$

$2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$\frac{\sqrt{3}}{2}$

13) $\sin\theta = \frac{3}{4}$



$m^2 + 3^2 = 4^2$

$m^2 + 9 = 16$

$m = \sqrt{7}$

$0^\circ \le \theta < 90^\circ$

$0^\circ \le 2\theta < 180^\circ$

2θ 2nd quad

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a. $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2}{1} \cdot \frac{3}{4} \cdot \frac{\sqrt{7}}{4} = \frac{6\sqrt{7}}{16} = \boxed{\frac{3\sqrt{7}}{8}}$

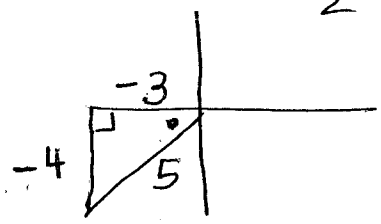
b. $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{3}{4}\right)^2 = 1 - \frac{1}{2} \cdot \frac{9}{16} = 1 - \frac{9}{8} = \boxed{-\frac{1}{8}}$

c. $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2 \cdot \frac{3}{\sqrt{7}}}{1 - \left(\frac{3}{\sqrt{7}}\right)^2} = \frac{\frac{6}{\sqrt{7}}}{1 - \frac{9}{7}} = \frac{\frac{6}{\sqrt{7}}}{-\frac{2}{7}}$

$= \frac{6^3}{\sqrt{7}} \cdot -\frac{7}{7_1} = \frac{-21}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{-21\sqrt{7}}{7}$

$= \boxed{-3\sqrt{7}}$

14) $\tan\theta = \frac{4}{3}$ $\quad \pi \le \theta \le \frac{3\pi}{2}$



$\frac{\pi}{2} \le \frac{\theta}{2} \le \frac{3\pi}{4}$

2nd quad

a. $\sin\frac{\theta}{2} = +\sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1 - -\frac{3}{5}}{2}} = \sqrt{\frac{\frac{8}{5}}{\frac{2}{1}}} = \sqrt{\frac{\cancel{8}^4}{5} \cdot \frac{1}{\cancel{2}_1}} = \boxed{\sqrt{\frac{4}{5}}}$

b. $\cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{1 + -\frac{3}{5}}{2}} = -\sqrt{\frac{\frac{2}{5}}{2}} = -\sqrt{\frac{2}{5} \cdot \frac{1}{2}} = \boxed{-\sqrt{\frac{1}{5}}}$

c. $\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{-\frac{4}{5}}{1 + -\frac{3}{5}} = \frac{-\frac{4}{5}}{\frac{2}{5}} = -\frac{4}{5} \cdot \frac{5}{2} = \boxed{-2}$