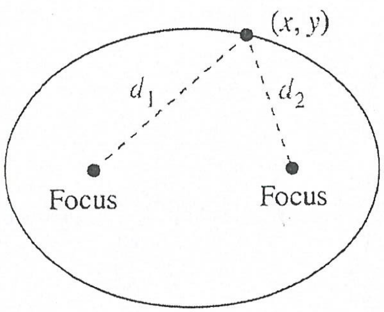
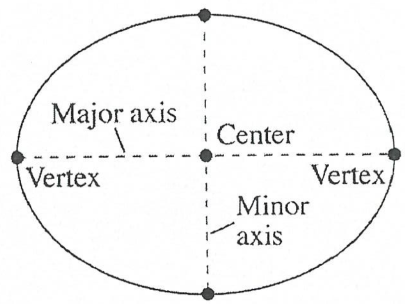
§9.2: Ellipses and Circles

Definition of an Ellipse

An **ellipse** is the set of all points (x, y) in a plane, the sum of whose distances from two distinct fixed points (**foci**) is constant. [See Figure 9.15(a).]



$d_1 + d_2$ is constant.



- The line through the foci intersects the ellipse at two points called **vertices**
- The chord joining the vertices is the **major axis**, and its midpoint is the **center** of the ellipse
  - Half the major axis is the *semi-major axis*
- The chord perpendicular to the major axis at the center is the **minor axis**
  - Half the minor axis is the *semi-minor axis*

The General Form of an Ellipse

- The general second-degree equation for *all* conics is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
- For ellipses, $B = 0$; A and C have the same sign
  - $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Standard Equation of an Ellipse

The **standard form of the equation of an ellipse** with center (h, k) and major and minor axes of lengths $2a$ and $2b$, respectively, where $0 < b < a$, is

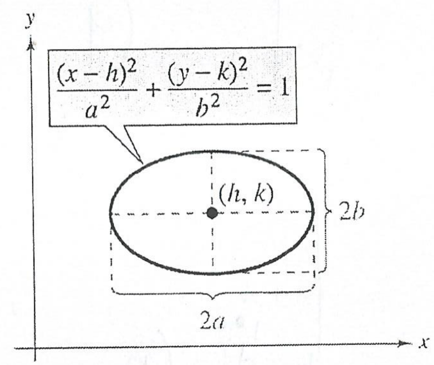
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

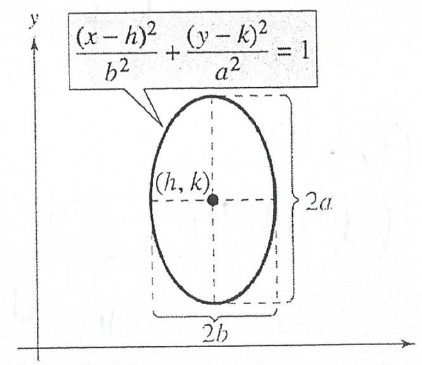
The foci lie on the major axis, c units from the center, with $c^2 = a^2 - b^2$. If the center is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad \text{Major axis is vertical.}$$



Major axis is horizontal.



Major axis is vertical.

Since an ellipse is clearly not a function, it can be written as a parametric function such that:

x = acosT + h or x = bcosT + h
y = bsinT + k or y = asinT + k
if the major axis is horizontal — if the major axis is vertical

Example 1

Convert $4x^2 + 9y^2 - 16x + 18y - 11 = 0$ into standard form.

$4x^2 - 16x \quad + 9y^2 + 18y \quad = 11$

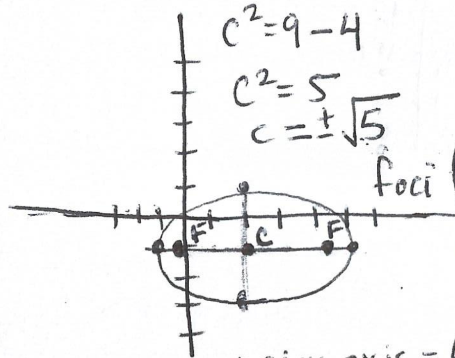
$4(x^2 - 4x + 4) + 9(y^2 + 2y + 1) = 11 + 16 + 9$

$\left(\frac{-4}{2}\right)^2 = 4 \qquad \left(\frac{2}{2}\right)^2 = 1$

$$\frac{4(x-2)^2 + 9(y+1)^2}{36} = \frac{36}{36}$$

$$\boxed{\frac{(x-2)^2}{9} + \frac{(y+1)^2}{4} = 1}$$

center $(2, -1)$



$c^2 = a^2 - b^2$

$c^2 = 9 - 4$

$c^2 = 5$

$c = \pm\sqrt{5}$

foci $(2-\sqrt{5}, -1)$

$(2+\sqrt{5}, -1)$

major axis = 6 units

minor axis = 4 units

$e = \frac{c}{a} = \frac{\sqrt{5}}{3} = .745$

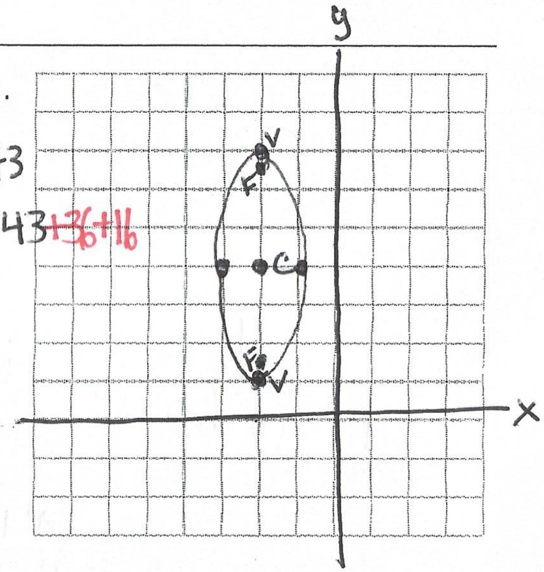
Graphing an Ellipse

- Convert the equation into standard form.
- Plot the center (h, k).
- If the major axis is parallel to the y-axis:
  - On the major axis, plot $(h \pm a, k)$.
  - On the minor axis, plot $(h, k \pm b)$
- If the major axis is parallel to the x-axis:
  - On the major axis, plot $(h, k \pm a)$.
  - On the minor axis, plot $(h \pm b, k)$
- Connect the outer points as a smooth ellipse.

- Always go a units in both directions on the major axis.
- Always go b units in both directions on the minor axis.

Example 2

Graph $9x^2 + y^2 + 36x - 8y + 43 = 0$.



$9x^2 + 36x \quad + y^2 - 8y \quad = -43$

$\left(\frac{4}{2}\right)^2 = 4$

$9(x^2 + 4x + 4) + y^2 - 8y + 16 = -43 + 36 + 16$

$\left(\frac{-8}{2}\right)^2 = 16$

$$\frac{9(x+2)^2 + (y-4)^2}{9} = \frac{9}{9}$$

$$\frac{(x+2)^2}{1} + \frac{(y-4)^2}{9} = 1$$

center $(-2, 4)$

$c^2 = a^2 - b^2$

$c^2 = 9 - 1$

$c^2 = 8$

$c = \pm\sqrt{8} = \pm 2\sqrt{2}$

foci $(-2, 4+2\sqrt{2})$

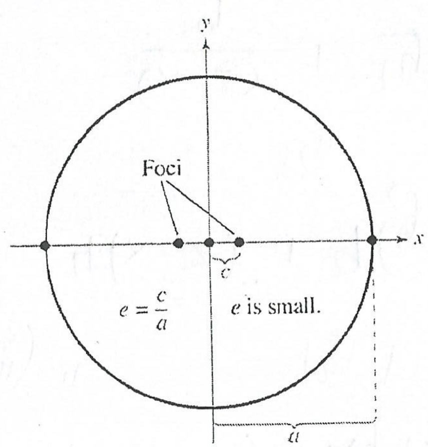
$(-2, 4-2\sqrt{2})$

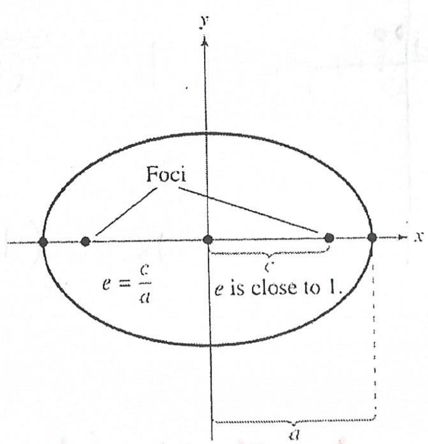
More on Ellipses

- The **foci** are located along the major axis between the vertices and the center. It follows that $0 < c < a$.
- Note that $c^2 = a^2 - b^2$. (This is <u>not</u> the same as the Pythagorean Theorem.)
- To measure the "ovalness" of an ellipse, you can use the concept of **eccentricity**.

> Definition of Eccentricity
>
> The **eccentricity** e of an ellipse is given by the ratio $e = \frac{c}{a}$.

- For ellipses: $0 < e < 1$.
- For an ellipse that is nearly circular, the value of e is closer to zero.
- For an elongated ellipse, the value of e is closer to one.





Example 3

What is the eccentricity of the ellipse from Example 2?

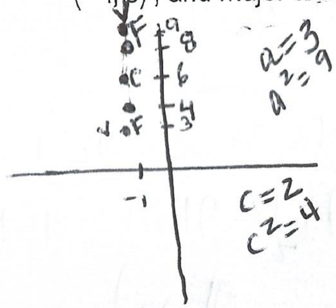
$$e = \frac{c}{a} = \frac{2\sqrt{2}}{3} = .943$$

close to 1

Example 4

Write the equation in standard form of an ellipse with foci at $(-1, 4)$ and $(-1, 8)$, and major axis of length 6. $2a = 6$

$a = 3$



$a = 3$
$a^2 = 9$

center $(-1, 6)$

$c = 2$
$c^2 = 4$

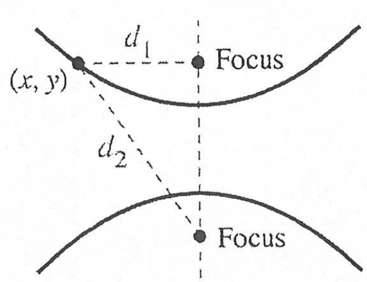
$$\boxed{\frac{(x+1)^2}{5} + \frac{(y-6)^2}{9} = 1}$$

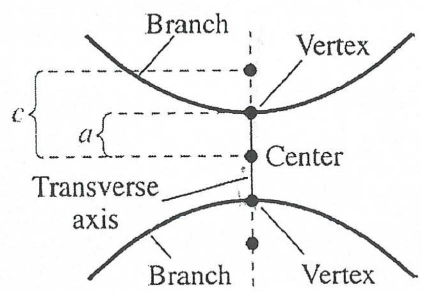
$c^2 = a^2 - b^2$
$4 = 9 - b^2$
$-5 = -b^2$
$b^2 = 5$

§9.3: Hyperbolas

Definition of a Hyperbola

A **hyperbola** is the set of all points (x, y) in a plane, the difference of whose distances from two distinct fixed points, the **foci**, is a positive constant.





$d_2 - d_1$ is a positive constant.

- The graph of a hyperbola has two disconnected parts (**branches**)
- The line through the two foci intersects the hyperbolas at two points (**vertices**)
- The line segment connecting the vertices is the **transverse axis**, and the midpoint of the transverse axis is the **center** of the hyperbola

The General Form of a Hyperbola

- The general second-degree equation for *all* conics is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.
- For ellipses, $B = 0$; A and C have different signs
  - $Ax^2 + Cy^2 + Dx + Ey + F = 0$

Standard Equation of a Hyperbola

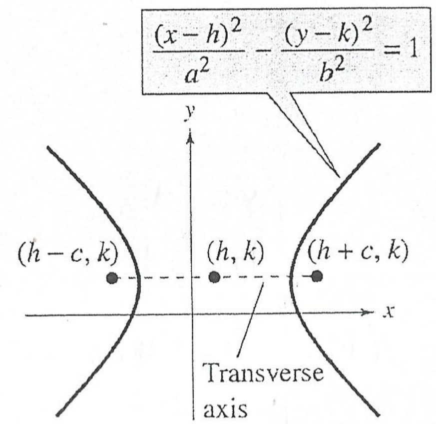
The **standard form of the equation of a hyperbola** with center at (h, k) is

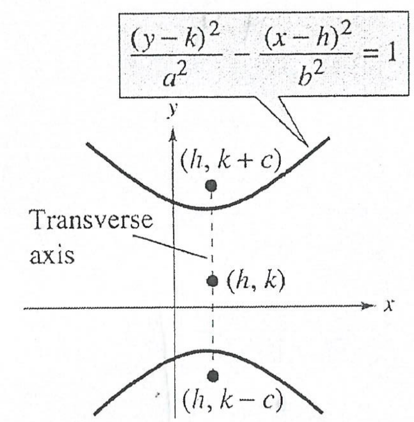
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.}$$

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \quad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center. Moreover, $c^2 = a^2 + b^2$. If the center of the hyperbola is at the origin $(0, 0)$, the equation takes one of the following forms.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{Transverse axis is horizontal.} \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \quad \text{Transverse axis is vertical.}$$





Example 1

Convert $-16x^2 + y^2 - 32x - 4y - 28 = 0$ into standard form.

$\left(\frac{2}{2}\right)^2 = 1$

$\left(\frac{-4}{2}\right)^2 = 4$

$-16x^2 - 32x \quad + y^2 - 4y \quad = 28$

$-16(x^2 + 2x + 1) + y^2 - 4y + 4 = 28 - 16 + 4$

$\frac{-16(x+1)^2 + (y-2)^2}{16} = \frac{16}{16}$

$(y-2)^2 - (x+1)^2 = 1$ center

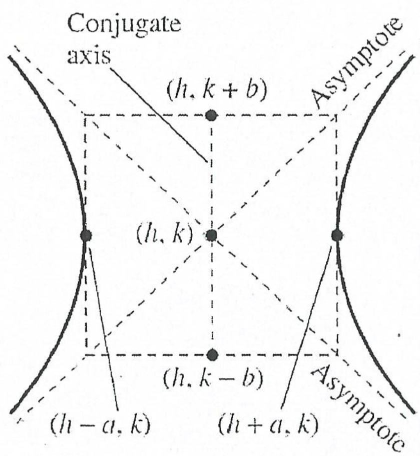
Asymptotes of a Hyperbola

- Each hyperbola has two **asymptotes** that intersect at the center of the hyperbola
- The asymptotes pass through the corners of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k)
- The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h, k+b)$ and $(h, k-b)$ if the transverse axis is horizontal
- The **conjugate axis** of a hyperbola is the line segment of length $2b$ joining $(h+b, k)$ and $(h-b, k)$ if the transverse axis is vertical

Asymptotes of a Hyperbola

$y = k \pm \frac{b}{a}(x-h)$ Asymptotes for horizontal transverse axis

$y = k \pm \frac{a}{b}(x-h)$ Asymptotes for vertical transverse axis



- The slopes for the asymptotes are always $\pm\frac{\text{rise}}{\text{run}}$
- The "run" always corresponds with the denominator of the x-term in the standard equation
  - If the transverse axis is horizontal, the hyperbola opens to the left/right, and the slopes are $\pm\frac{b}{a}$ because $\frac{(x-h)^2}{a^2}$ in the standard equation
  - If the transverse axis is vertical, the hyperbola opens up/down, and the slopes are $\pm\frac{a}{b}$ because $\frac{(x-h)^2}{b^2}$ in the standard equation

Graphing a Hyperbola

- Convert the equation into standard form.
- Plot the center (h, k).
- If the transverse axis is parallel to the x-axis:
  - Plot the vertices $(h \pm a, k)$.
- If the transverse axis is parallel to the y-axis:
  - Plot the vertices $(h, k \pm a)$.
- Determine the equations of the asymptotes.
- Plot the asymptotes.
- Draw the hyperbola.

Example 2

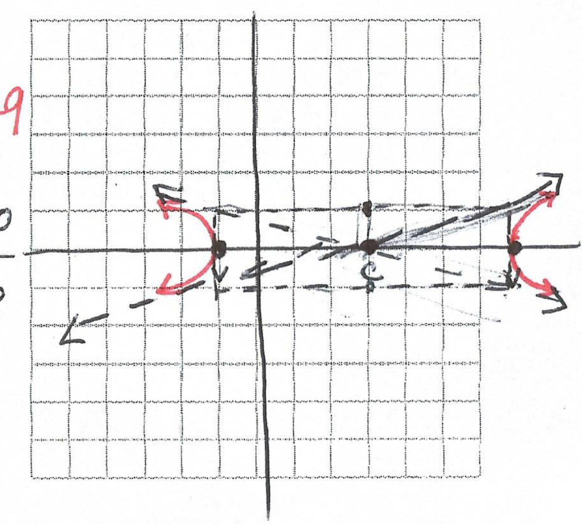
Graph $x^2 - 16y^2 - 6x - 7 = 0$.

$x^2 - 6x + 9 - 16y^2 = 7 + 9$

$\left(-\frac{6}{2}\right)^2 = 9$ $\frac{(x-3)^2 - 16y^2}{16} = \frac{16}{16}$

$\frac{(x-3)^2}{16} - \frac{y^2}{1} = 1$

center $(3,0)$



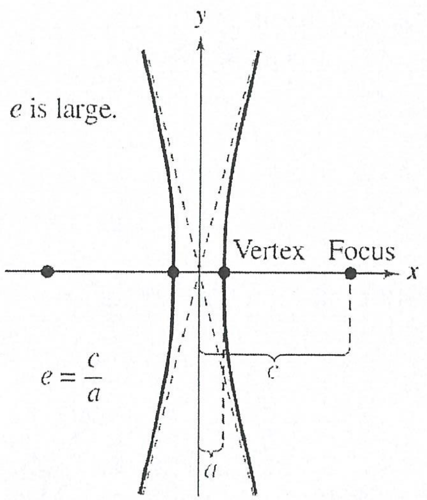
asymptotes

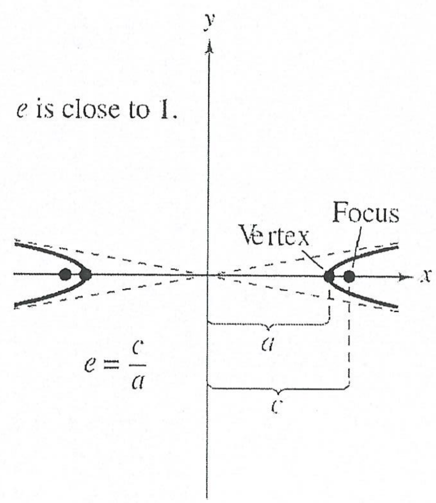
$y = 0 \pm \frac{1}{4}(x-3)$

$c^2 = a^2 + b^2$
$c^2 = 16 + 1$
$c^2 = 17$
$c = \sqrt{17}$

More on Hyperbolas

- The **foci** are located along the transverse axis. It follows that $0 < a < c$.
- Note that $c^2 = a^2 + b^2$. (This is <u>not</u> the same as with the ellipse.)
- As with ellipses, the **eccentricity** of a hyperbola is $e = \dfrac{c}{a}$.
- For hyperbolas, $e > 1$.
- If the eccentricity is large, the branches of the hyperbola are nearly flat.
- If the eccentricity is close to 1, the branches of the hyperbola are more pointed.





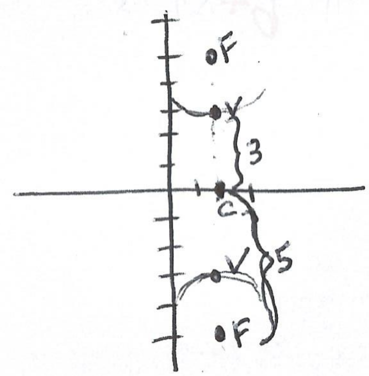
Example 3

What is the eccentricity of the hyperbola from Example 2?

$$e = \frac{c}{a} = \frac{\sqrt{17}}{4} = 1.031$$

Example 4

Write the equation in standard form of a hyperbola with foci $(2, 5)$ and $(2, -5)$, and vertices $(2, 3)$ and $(2, -3)$.



center $(2, 0)$

$$\boxed{\frac{(y-0)^2}{9} - \frac{(x-2)^2}{16} = 1}$$

$c^2 = a^2 + b^2$

$(5)^2 = 9 + b^2$

$25 = 9 + b^2$

$16 = b^2$

Classification of Conics

- Using the general form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$:

Conic	Test	Notes
Circle	$A = C$	$A \neq 0$
Parabola	$AC = 0$	$A = 0$ or $C = 0$, but not both
Ellipse	$AC > 0$	A and C have the same signs
Hyperbola	$AC < 0$	A and C have different signs

Example 5

Classify each of the following without converting to standard form:

a) $4x^2 + 5y^2 - 9x + 8y = 0$ ellipse

b) $2x^2 - 5x + 7y - 8 = 0$ parabola

c) $7x^2 + 7y^2 - 9x + 8y - 16 = 0$ circle

d) $4x^2 - 5y^2 - x + 8y + 1 = 0$ hyperbola

Summarizing Eccentricity

Interval	Curve	c	e	Note
$e = 0$	circle	0	0	
$0 < e < 1$	ellipse	$\sqrt{a^2 - b^2}$	c/a	Closer to 0—more circular Closer to 1—more elongated
$e = 1$	parabola	p	1	
$e > 1$	hyperbola	$\sqrt{a^2 + b^2}$	c/a	Closer to 1—more pointed Larger values—more flat