AP Calculus BC

Notes 3.6: Error

Actual Error

This is the real amount of error, not the error bound (worst case scenario). It is the difference between the actual f(x) and the polynomial.

Steps:

1. plug x into f(x)
2. plug x into the polynomial
3. error = |difference between the two|

Ex 1) Approximate f(.1) using a 2nd degree Maclaurin polynomial for $f(x) = \frac{1}{1-x}$. $= \sum_{n=0}^{\infty} x^n$

$P_2(x) = 1 + x + x^2$

$P_2(.1) = 1 + .1 + .1^2 = 1.11$

$f(.1) = 1.\bar{1} = \frac{10}{9}$

actual error $= \left|1.11 - \frac{10}{9}\right| = .00\bar{1}$
$= \frac{1}{900}$

Error Bound

- Gives the largest possible error in an estimate. The actual error will be less than the bound.
- The two types of error bounds we use are alternating series & Lagrange.

Alternating series

If a series is alternating and decreasing, the error bound (worst case scenario) can be found by taking the absolute value of the n+1 term.

(next omitted term)

Ex 2) Find the error bound of the approximation of $F(x)$.

$$F(x) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{(-1)^{n+1}}{n}$$

$$F(x) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}$$

e.b. $= \left|-\frac{1}{6}\right| = \frac{1}{6}$

error $< \frac{1}{6}$

Ex 3) Use the alternating harmonic series. Find a bound for the truncation error after 99 terms.

$$\sum_{n=1}^{99} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

$$\text{error} < |100^{th}\text{ term}|$$

$$-\frac{1}{100}$$

$$\text{error} < \frac{1}{100}$$

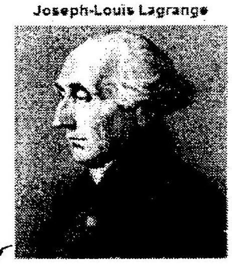
Ex 4) Given $\sum \frac{(-1)^{n+1}}{n^6}$, how many terms are to be added to get the |error| to be < 0.00005?

n	$\lvert a_n \rvert$
1	$\left\lvert \frac{1}{1^6} \right\rvert = 1$
2	$\left\lvert \frac{-1}{2^6} \right\rvert = .015625$
3	$\left\lvert \frac{1}{3^6} \right\rvert = .00137$
4	$\left\lvert \frac{-1}{4^6} \right\rvert = .000244$
5	$\left\lvert \frac{1}{5^6} \right\rvert = .000064$
6	$\left\lvert \frac{-1}{6^6} \right\rvert = .0000214$

between

5 terms

Joseph-Louis Lagrange



Lagrange

This method uses a special form of the Taylor formula to find the **error bound** of a polynomial approximation of a Taylor Series.

max value

The Lagrange Formula:

$$e.b. = \left| \frac{f^{(n+1)}(z)}{(n+1)!} (x-a)^{n+1} \right|$$

center

The variable z is a number between x and a, but to find the error bound, z ends up being equal to one of the two. To determine whether the z value will be the same as x or a, you must plug each number into $|f^{(n+1)}(z)|$ to see which gives the greatest number.

For example: If you are trying to find the error of a 2nd degree Taylor polynomial approximation of $f(x) = \frac{1}{1-x}$, you must find the 3rd derivative because the formula uses $f^{(n+1)}(z)$.

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$$

$$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$$

$$f'''(x) = -6(1-x)^{-4}(-1) = \frac{6}{(1-x)^4}$$

refers to ex 1

For this function, x=.1 and a=0. Plug these two values into the ________ derivative.

$$f'''(.1) = 9.145 \leftarrow \text{bigger}$$

$$f'''(0) = 6$$

Plug the larger of the two into $f^{(n+1)}(z)$ in the Lagrange formula to find the error bound.

$$e.b. = \left| \frac{9.145}{3!} (.1 - 0)^3 \right| = .00152$$

Exception! When $f(x) = \sin x$ or $\cos x$, the value for $f^{(n+1)}(z)$ will always be equal to 1, because that is the greatest value of $\sin x$ or $\cos x$.

$\rightarrow n=5$

Ex 5) Find the upper bound for the error for the 5th degree polynomial approximation of e^1.

know $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \ldots$

6th deriv $= e^x$

$a=0 \quad x=1$

$f^{(6)}(0) = e^0 = 1$

$f^{(6)}(1) = e^1 = e \leftarrow$ bigger $z=1$

$$\text{e.b.} = \left| \frac{e}{6!}(1-0)^6 \right| = \frac{e}{6!}$$

$.00378$

Ex 6) a) Find the 3rd degree Maclaurin polynomial for $f(x) = \sin x$.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \boxed{x - \frac{x^3}{3!}} + \ldots$$

b) Approximate $\sin(.1)$ using the polynomial.

$$.1 - \frac{.1^3}{3!} = .0998333$$

c) Determine the accuracy of the approximation (find the Lagrange error bound).

$$\text{e.b.} = \left| \frac{1}{(3+1)!}(.1-0)^4 \right| = .000004167$$