Integration Review
(All Methods)

1. $\int_0^{\infty} \frac{-4}{x^2+11x+30}\,dx = \lim_{b\to\infty} \int_0^b \left(\frac{-4}{x+5} + \frac{4}{x+6}\right)dx$

$\frac{-4}{(x+5)(x+6)} = \frac{A}{x+5} + \frac{B}{x+6}$

$-4 = A(x+6) + B(x+5)$

if $x=-6$: $B = 4$

$x = 5$: $A = -4$

$= \lim_{b\to\infty} \left(-4\ln|x+5| + 4\ln|x+6| + C \Big|_0^b\right)$

$= \lim_{b\to\infty}\left(\underbrace{-4\ln|b+5| + 4\ln|b+6|} - (-4\ln 5 + 4\ln 6)\right)$

$= \lim_{b\to\infty}\left(4\ln\left|\frac{b+6}{b+5}\right| + 4\ln\left(\frac{5}{6}\right)\right)$

$= 4\cdot\ln 1 + 4\ln\left(\frac{5}{6}\right) = \boxed{4\ln\left(\frac{5}{6}\right)}$

2. $\int \frac{2x}{x^2-3x-10}\,dx = \int \left(\frac{10/7}{x-5} + \frac{4/7}{x+2}\right)dx$

$\frac{2x}{(x-5)(x+2)} = \frac{A}{x-5} + \frac{B}{x+2}$

$2x = A(x+2) + B(x-5)$

if $x = -2$: $-4 = -7B$

$B = 4/7$

$x = 5$: $10 = 7A$

$A = 10/7$

$= \boxed{\frac{10}{7}\ln|x-5| + \frac{4}{7}\ln|x+2| + C}$

3. $\int \frac{1}{\sqrt{x}\,(1+\sqrt{x})^2}\,dx$

$u = 1+\sqrt{x}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$2\,du = \frac{dx}{\sqrt{x}}$

$2\int u^{-2}\,du = -2u^{-1} + C$

$= -2\left(1+\sqrt{x}\right)^{-1} + C$

$= \boxed{\frac{-2}{1+\sqrt{x}} + C}$

4. $\int_1^\infty x e^{-5x}\,dx$

	u	dv
$+$	x	e^{-5x}
$-$	1	$-\frac{1}{5}e^{-5x}$
$+$	0	$\frac{1}{25}e^{-5x}$

$$\lim_{b\to\infty}\int_1^b xe^{-5x}\,dx = \lim_{b\to\infty}\left(-\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C\right)\Big|_1^b$$

$$= \lim_{b\to\infty}\left(\frac{-b}{5e^{5b}} - \frac{1}{25e^{5b}} - \left(-\frac{1}{5}e^{-5} - \frac{1}{25}e^{-5}\right)\right)$$

$$= 0 - 0 + \frac{1}{5}e^{-5} + \frac{1}{25}e^{-5} = \boxed{\frac{6}{25e^5}}$$

5. $\lim_{x\to\infty}\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)^x \quad 1^\infty$

$\lim_{x\to\infty} x\cdot\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right) \quad \infty\cdot 0$

$\lim_{x\to\infty} \dfrac{\ln\left(1 + \frac{3}{x} + \frac{5}{x^2}\right)}{\frac{1}{x}} \quad \frac{0}{0}$

$$\lim_{x\to\infty} \frac{\left(\frac{1}{1 + \frac{3}{x} + \frac{5}{x^2}}\right)\left(-\frac{3}{x^2} - \frac{10}{x^3}\right)(-x^2)}{\left(-\frac{1}{x^2}\right)(-x^2)} = \lim_{x\to\infty}\frac{3 + \frac{10}{x}}{1 + \frac{3}{x} + \frac{5}{x^2}} = \frac{3}{1} = 3$$

$\boxed{e^3}$

6. $$\int \frac{x - x^2}{2\sqrt[3]{x}}\,dx = \frac{1}{2}\int\left(x^{2/3} - x^{5/3}\right)dx = \boxed{\frac{1}{2}\left(\frac{3}{5}x^{5/3} - \frac{3}{8}x^{8/3} + C\right)}$$

7. $\displaystyle\int \frac{dx}{(x+3)(x-2)} = \int\left(\frac{-1/5}{x+3} + \frac{1/5}{x-2}\right)dx$

$\dfrac{A}{x+3} + \dfrac{B}{x-2} = \dfrac{1}{(x+3)(x-2)}$

$A(x-2) + B(x+3) = 1$

if $x=2$: $5B = 1$ $\quad B = 1/5$

$x=-3$ $\quad -5A = 1$ $\quad A = -1/5$

$$= \boxed{-\tfrac{1}{5}\ln|x+3| + \tfrac{1}{5}\ln|x-2| + C \quad \text{OR} \quad \tfrac{1}{5}\ln\left|\frac{x-2}{x+3}\right| + C}$$

8. $\displaystyle\int \ln\left(x^{157}\right)dx = 157\int \ln x\,dx$

$u = \ln x \quad dv = dx$

$\rightarrow\ du = \frac{1}{x}dx \quad v = x$

$x\ln x - \int x\cdot\frac{1}{x}dx$

$x\ln x - x + C$

$$\boxed{157\left(x\ln x - x + C\right)}$$

9. $\displaystyle\int \frac{x^2}{(1-x^3)^2}\,dx$

$u = 1-x^3$

$\frac{du}{dx} = -3x^2 \qquad -\frac{1}{3}du = x^2dx$

$$-\frac{1}{3}\int u^{-2}du = \frac{1}{3}u^{-1} + C = \frac{1}{3}(1-x^3)^{-1} + C = \boxed{\frac{1}{3(1-x^3)} + C}$$

10. $\displaystyle\int_0^9 \frac{dx}{(x-9)^{2/3}} = \lim_{b\to 9^-}\int_0^b \frac{dx}{(x-9)^{2/3}}$

$u = x-9$

$\frac{du}{dx} = 1 \quad du = dx$

$\displaystyle\int \frac{du}{u^{2/3}} = \int u^{-2/3}du = 3u^{1/3} + C$

$$\lim_{b\to 9^-}\left(3(x-9)^{1/3} + C\,\Big|_0^b\right)$$

$$= \lim_{b\to 9^-}\left(3(b-9)^{1/3} - \left(3\cdot(-9)^{1/3}\right)\right) = 0 + 3\cdot\sqrt[3]{9} = \boxed{3\sqrt[3]{9}}$$

11. $\lim_{x \to 0} \frac{\tan px}{\tan qx} \quad \frac{0}{0}$

$\lim_{x \to 0} \frac{p \cdot \sec^2 px}{q \cdot \sec^2 qx} = \boxed{\frac{p}{q}}$

12. $\int_{-2}^{2} \frac{1}{x}\, dx$ discont at $x=0$ $\quad \int_{-2}^{0} \frac{1}{x} dx + \int_{0}^{2} \frac{1}{x} dx$

$\lim_{p \to 0^-} \int_{-2}^{p} \frac{1}{x} dx + \lim_{p \to 0^+} \int_{p}^{2} \frac{1}{x} dx$

$\lim_{p \to 0^-} \left(\ln|x| + c \Big|_{-2}^{p} \right) + \lim_{p \to 0^+} \left(\ln|x| + c \Big|_{p}^{2} \right)$

$\lim_{p \to 0^-} \left(\ln|p| - \ln 2 \right) + \lim_{p \to 0^+} \left(\ln 2 - \ln|p| \right)$

$\boxed{\text{diverges}}$

13. $\lim_{x \to 1} \left(\frac{\ln x^2}{x^2 - 1} \right) \quad \frac{0}{0} \qquad \lim_{x \to 1} \frac{\frac{1}{x^2} \cdot 2x}{2x} = \lim_{x \to 1} \frac{1}{x^2} = \boxed{1}$

14. $\int x \sqrt{x-5}\, dx \qquad u = x-5 \longrightarrow x = u+5$

$\frac{du}{dx} = 1 \quad du = dx$

$\int u^{1/2}(u+5)\, du = \int \left(u^{3/2} + 5u^{1/2} \right) du = \frac{2}{5} u^{5/2} + \frac{10}{3} u^{3/2} + C$

$= \boxed{\frac{2}{5}(x-5)^{5/2} + \frac{10}{3}(x-5)^{3/2} + C}$

15. $\int x^2 \sin 2x \, dx$

	u	dv
$+$	x^2	$\sin 2x$
$-$	$2x$	$-\frac{1}{2}\cos 2x$
$+$	2	$-\frac{1}{4}\sin 2x$
$-$	0	$\frac{1}{8}\cos 2x$

$$\boxed{-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C}$$

16. $$\lim_{x\to\infty} x\cdot e^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \boxed{0}$$

17. $$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \quad 1^\infty$$

$$\lim_{n\to\infty} n\cdot \ln\left(1+\frac{1}{n}\right) \quad \infty\cdot 0$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} \quad \frac{0}{0}$$

$$\lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}}\cdot -\frac{1}{n^2}}{-\frac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1 \quad \boxed{e^1}$$

18. $$\int \frac{x}{\sqrt{1-x^2}}\,dx \qquad u = 1-x^2$$

$$\frac{du}{dx} = -2x \qquad -\frac{1}{2}du = x\,dx$$

$$-\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}\,du = -1u^{1/2} + C$$

$$\boxed{-\sqrt{1-x^2} + C}$$