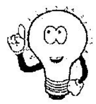


When dealing with a **RELATION** that is not a function, it is often possible to solve for y. Then you can identify the functions which are **IMPLICITLY** defined by the original relation.

***HINT: SOLVE FOR Y. If the equation starts with y^2 when solving you will always end with ± some expression. The positive (+) is one equation, and the negative (-) is the other. These two different equations are the two implicitly defined equations for the given relation.

Example 1 Find two functions defined implicitly by each given relation.

a) $x = y^2$

$y = \pm\sqrt{x}$

b) $x^2 + 2xy + y^2 = 1$

$(x+y)(x+y) = 1$

$\sqrt{(x+y)^2} = \pm\sqrt{1}$

$x + y = \pm 1$

$y = -x \pm 1$

$y = -x + 1$

$y = -x - 1$

c) $9x^2 - 12xy + 4y^2 = 16$

$(3x-2y)(3x-2y) = 16$

$\sqrt{(3x-2y)^2} = \pm\sqrt{16}$

$3x - 2y = \pm 4$

$-2y = -3x \pm 4$

$y = \dfrac{-3x \pm 4}{-2}$

$y = \dfrac{-3x+4}{-2} = \frac{3}{2}x - 2$

$y = \dfrac{-3x-4}{-2} = \frac{3}{2}x + 2$

d) $7x^2 - 14xy - 63 = -7y^2$

$7x^2 - 14xy + 7y^2 = 63$

$7(x^2 - 2xy + y^2) = 63$

$7(x-y)(x-y) = 63$

$7(x-y)^2 = 63$

$\sqrt{(x-y)^2} = \pm\sqrt{9}$

$x - y = \pm 3$

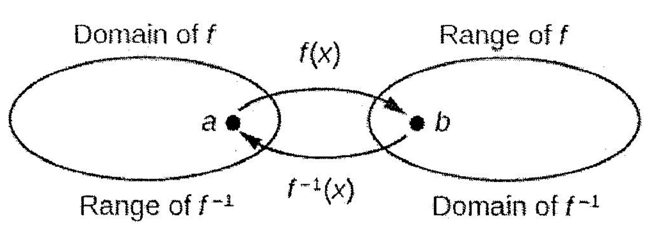
$-y = -x \pm 3$

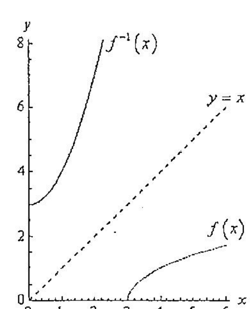
$y = \dfrac{-x \pm 3}{-1}$

$y = x - 3$ $y = x + 3$

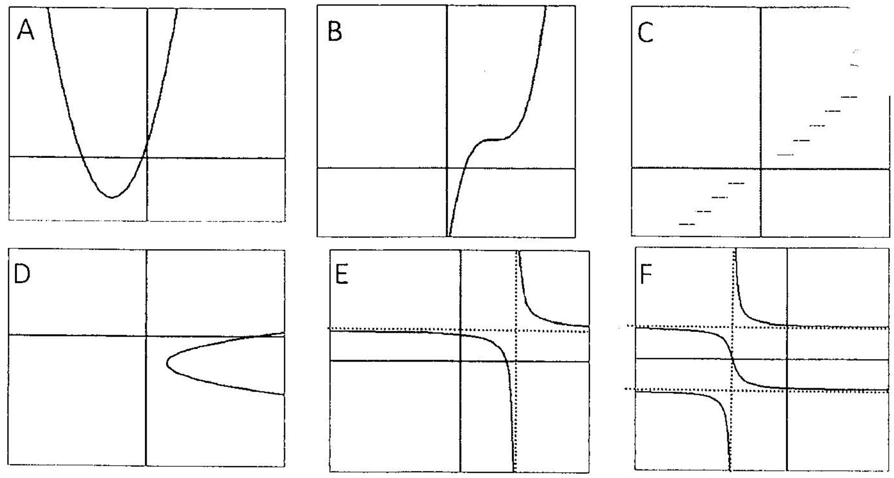
Inverse Functions & Relations

- The most important thing to remember about INVERSES is that x & y switch.
- The inverse of $f(x)$ is denoted $f^{-1}(x)$
- If $f(x)$ & $g(x)$ are inverses of one another then the domain of one is the range of the other & vice-versa.
- A relation is a function if it passes the Vertical Line Test (VLT)
- A relation has an inverse that is a function if it passes the Horizontal Line Test (HLT)
- A function has an inverse function if it is a one-to-one function (meaning it passes both the HLT & VLT)
- The graphical relationship between inverses is that they are reflections of one another over the line $y = x$





Using the Vertical & Horizontal Line Tests



Example 2

(a) Which of the relations to the right are functions? pass VLT

A, B, C, E

(b) Which of the relations to the right have an inverse function? pass HLT

B, D, E, F

(c) Which of the relations are one-to-one functions?

B, E

Calculating Inverses Algebraically

Example 3 Given the function $f(x)$ calculate $f^{-1}(x)$ and identify the domain and range of each:

(a) $f(x) = 3\sqrt{x} - 5$ $\qquad f^{-1}(x) = \dfrac{(x+5)^2}{9}$

Domain of $f(x)$: $[0, \infty)$ Domain of $f^{-1}(x)$: $[-5, \infty)$

Range of $f(x)$: $[-5, \infty)$ Range of $f^{-1}(x)$: $[0, \infty)$

$x = 3\sqrt{y} - 5$

$x + 5 = 3\sqrt{y}$

$\left(\dfrac{x+5}{3}\right)^2 = \left(\sqrt{y}\right)^2$

$y = \left(\dfrac{x+5}{3}\right)^2 = \dfrac{(x+5)^2}{9}$

(b) $f(x) = \dfrac{x-4}{x+3}$ $\qquad f^{-1}(x) = \dfrac{3x+4}{1-x}$

Domain of $f(x)$: $(-\infty, -3) \cup (-3, \infty)$ Domain of $f^{-1}(x)$: $(-\infty, 1) \cup (1, \infty)$

Range of $f(x)$: $(-\infty, 1) \cup (1, \infty)$ Range of $f^{-1}(x)$: $(-\infty, -3) \cup (-3, \infty)$

$\dfrac{x}{1} = \dfrac{y-4}{y+3}$

$x(y+3) = 1(y-4)$

$xy + 3x = y - 4$

$3x + 4 = y - xy$

$3x + 4 = y(1-x)$

$\dfrac{3x+4}{1-x} = y$

(c) $f(x) = -(x+1)^3 - 5$ $\qquad f^{-1}(x) =$ $-1+\sqrt[3]{-x-5}$

Domain of $f(x)$: $(-\infty, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, \infty)$

Range of $f(x)$: $(-\infty, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, \infty)$

$x = -(y+1)^3 - 5$

$x+5 = -(y+1)^3$

$-x-5 = (y+1)^3$

$\sqrt[3]{-x-5} = y+1$

$-1+\sqrt[3]{-x-5} = y$

(d) $f(x) = \dfrac{3}{x-5}$ $\qquad f^{-1}(x) =$ $5+\dfrac{3}{x}$

Domain of $f(x)$: $(-\infty, 5) \cup (5, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, 0) \cup (0, \infty)$

Range of $f(x)$: $(-\infty, 0) \cup (0, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, 5) \cup (5, \infty)$

$\dfrac{x}{1} = \dfrac{3}{y-5}$

$x(y-5) = 3$

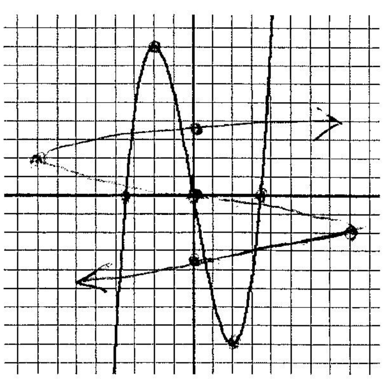
$xy - 5x = 3$

$xy = 5x+3$

$y = \dfrac{5x+3}{x} = 5+\dfrac{3}{x}$

Sketching an Inverse Relation From a Graph

Example 4 Given the function $f(x)$ below, sketch $f^{-1}(x)$ and identify the domain and range of both.



D of $f(x)$: $(-\infty, \infty)$ $\qquad$ **R** of $f(x)$: $(-\infty, \infty)$

D of $f^{-1}(x)$: $(-\infty, \infty)$ $\qquad$ **R** of $f^{-1}(x)$: $(-\infty, \infty)$

The Inverse Composition Rule

A function f is one-to-one with inverse function g if and only if $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f.

Verifying Inverses

Example 5 Given the two functions below, verify that they are inverses of one another.

(a) $f(x) = -\frac{1}{2}(x+3)^2 - 4$ & $g(x) = \sqrt{-2x-8} - 3$

$$f(g(x)) = -\frac{1}{2}\left(\sqrt{-2x-8} - 3 + 3\right)^2 - 4 = -\frac{1}{2}\left(\sqrt{-2x-8}\right)^2 - 4$$
$$= -\frac{1}{2}(-2x-8) - 4 = x + 4 - 4 = x$$

$$g(f(x)) = \sqrt{-2\left(-\frac{1}{2}(x+3)^2 - 4\right) - 8} - 3 = \sqrt{(x+3)^2 + 8 - 8} - 3$$
$$= \sqrt{(x+3)^2} - 3 = x + 3 - 3 = x$$

(b) $f(x) = \frac{x-11}{3}$ & $g(x) = 3x + 11$

$$f(g(x)) = \frac{3x + 11 - 11}{3} = \frac{3x}{3} = x$$

$$g(f(x)) = 3\left(\frac{x-11}{3}\right) + 11 = x - 11 + 11 = x$$

(c) $f(x) = 5\sqrt[3]{x} - 7$ & $g(x) = \frac{(x+7)^3}{125}$

$$f(g(x)) = 5\sqrt[3]{\frac{(x+7)^3}{125}} - 7 = 5 \cdot \frac{x+7}{5} - 7 = x + 7 - 7 = x$$

$$g(f(x)) = \frac{(5\sqrt[3]{x} - 7 + 7)^3}{125} = \frac{(5\sqrt[3]{x})^3}{125} = \frac{125x}{125} = x$$