When dealing with a **RELATION** that is NOT a ***FUNCTION*** it is often possible to solve for y, then identifying the functions which **IMPLICITLY** defined by the original relation.

HINT: SOLVE FOR Y! If the equation starts with y^2 when solving you will always end with ± some expression. The positive (+) is one equation and the negative (-) is the other. These two different equations are the two implicitly defined equations for the relation given.

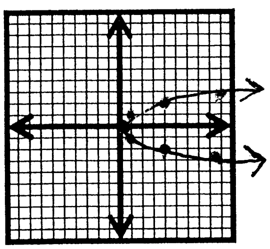
Graph of each of the relations below by determining the functions which are implicitly defined within them.

a) $x = y^2$

$\pm\sqrt{x} = y$

$y = \sqrt{x}$

$y = -\sqrt{x}$



b) $x^2 + 2xy + y^2 = 1$

$(x+y)(x+y) = 1$

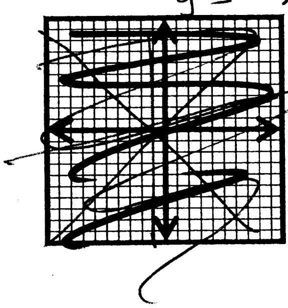
$\sqrt{(x+y)^2} = \pm\sqrt{1}$

$x + y = \pm 1$

$y = -x \pm 1$

$y = -x + 1$

$y = -x - 1$



c) $9x^2 - 12xy + 4y^2 = 16$

$(3x-2y)(3x-2y) = 16$

$\sqrt{(3x-2y)^2} = \pm\sqrt{16}$

$3x - 2y = \pm 4$

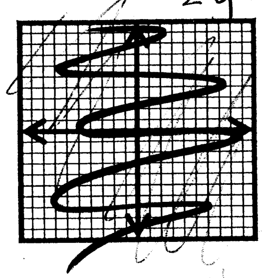
$-2y = -3x \pm 4$

$y = \dfrac{-3x \pm 4}{-2}$

$y = \frac{3}{2}x \pm -2$

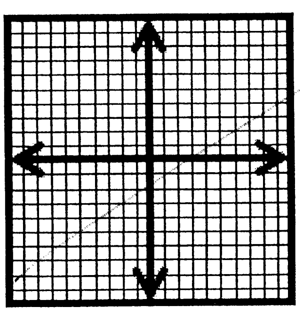
$y = \frac{3}{2}x - 2$

$y = \frac{3}{2}x + 2$

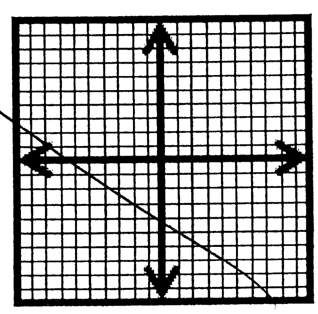


Now You Try:

d) $3x^2 + 75y^2 = 27 - 30xy$



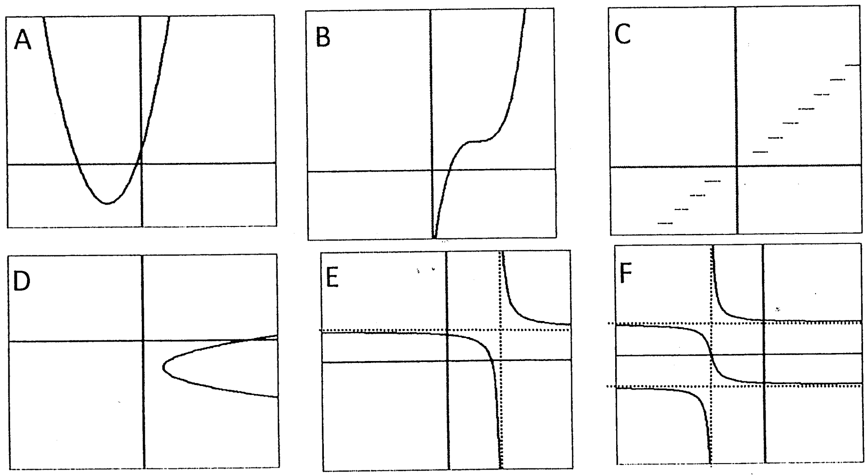
e) $7x^2 - 14xy - 63 = -7y^2$



Inverse Functions & Relations

- The most important thing to remember about INVERSES is that x & y switch.
- The inverse of $f(x)$ is denoted $f^{-1}(x)$
- If $f(x)$ & $g(x)$ are inverses of one another then the domain of one is the range of the other & vice-versa.
- A relation is itself a function if it passes the Vertical Line Test (VLT)
- A relation has an inverse that is a function if it passes the Horizontal Line Test (HLT)
- A function has an inverse function if it is a one-to-one function (meaning it passes both the HLT & VLT)
- The graphical relationship between inverses is that they are reflections of one another over the line $y = x$

Using the Vertical & Horizontal Line Tests



(a) Which of the relations to the right are functions? (VLT)

A, B, C, E

(b) Which of the relations to the right have an inverse function? (HLT)

B, D, E, F

(c) Which of the relations are ~~on~~ one-to-one functions?

B, E

Calculating Inverses Algebraically

Given the function $f(x)$ calculate $f^{-1}(x)$ and identify the domain and range of each:

(a) $\overset{y}{f(x)} = 3\sqrt{x} - 5$ $\qquad f^{-1}(x) = \left(\frac{x+5}{3}\right)^2$

Domain of $f(x)$: $[0, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $[-5, \infty)$

Range of $f(x)$: $[-5, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $[0, \infty)$

$x = 3\sqrt{y} - 5$

$\frac{x+5}{3} = \frac{3\sqrt{y}}{3}$ $\qquad y = \left(\frac{x+5}{3}\right)^2$

(b) $f(x) = \frac{x-4}{x+3}$ $\qquad f^{-1}(x) = \frac{3x+4}{1-x}$

Domain of $f(x)$: $(-\infty, -3) \cup (-3, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, 1) \cup (1, \infty)$

Range of $f(x)$: $(-\infty, 1) \cup (1, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, -3) \cup (-3, \infty)$

$\frac{x}{1} = \frac{y-4}{y+3}$

$y - 4 = x(y+3)$

$y - 4 = xy + 3x$

$y - xy = 3x + 4$

$\frac{y(1-x)}{1-x} = \frac{3x+4}{1-x}$

Now You Try

(c) $f(x) = -(x+1)^3 - 5$ $\qquad f^{-1}(x) = \sqrt[3]{-x-5} - 1$

Domain of $f(x)$: $(-\infty, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, \infty)$

Range of $f(x)$: $(-\infty, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, \infty)$

$x = -(y+1)^3 - 5$

$\frac{x+5}{-1} = \frac{-(y+1)^3}{-1}$

$\sqrt[3]{-x-5} = \sqrt[3]{(y+1)^3}$

$\sqrt[3]{-x-5} = y + 1$ $\qquad y = \sqrt[3]{-x-5} - 1$

(d) $f(x) = \frac{3}{x-5}$ $\qquad f^{-1}(x) = \frac{5x+3}{x} = 5 + \frac{3}{x}$

Domain of $f(x)$: $(-\infty, 5) \cup (5, \infty)$ $\qquad$ Domain of $f^{-1}(x)$: $(-\infty, 0) \cup (0, \infty)$

Range of $f(x)$: $(-\infty, 0) \cup (0, \infty)$ $\qquad$ Range of $f^{-1}(x)$: $(-\infty, 5) \cup (5, \infty)$

$\frac{x}{1} = \frac{3}{y-5}$

$x(y-5) = 3$

$xy - 5x = 3$

$xy = 5x + 3$

$y = \frac{5x+3}{x}$

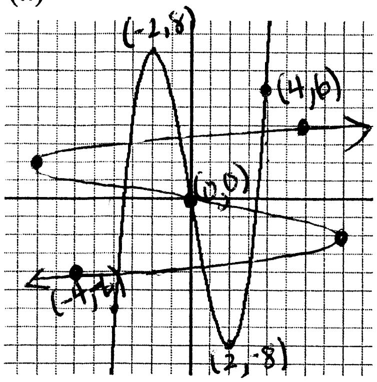
$x(y-5) = 3$

$y - 5 = \frac{3}{x}$

$y = 5 + \frac{3}{x}$

Sketching an Inverse Relation From a Graph
Given the function $f(x)$ below sketch $f^{-1}(x)$ and identify the domain and range of both.

(a)



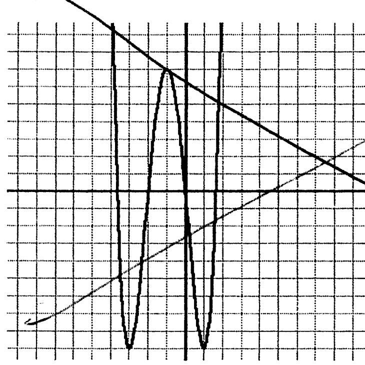
D of $f(x)$: $(-\infty,\infty)$

R of $f(x)$: $(-\infty,\infty)$

D of $f^{-1}(x)$: $(-\infty,\infty)$

R of $f^{-1}(x)$: $(-\infty,\infty)$

(b) NOW YOU TRY ☺



D of $f(x)$: ________

R of $f(x)$: ________

D of $f^{-1}(x)$: ________

R of $f^{-1}(x)$: ________

The Inverse Composition Rule
A function f is one-to-one with inverse function g if and only if $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f.

Verifying Inverses
Given the two functions below verify

(a) $f(x) = -\frac{1}{2}(x+3)^2 - 4$ & $g(x) = \sqrt{-2x-8} - 3$

$$f(g(x)) = -\frac{1}{2}\left(\sqrt{-2x-8} \cancel{-3} \cancel{+3}\right)^2 - 4$$
$$= -\frac{1}{2}(-2x-8) - 4$$
$$= x \cancel{+4} \cancel{-4}$$
$$= x$$

(b) $f(x) = \frac{x-11}{3}$ & $g(x) = 3x+11$

$$f(g(x)) = \frac{3x \cancel{+11} \cancel{-11}}{3} = \frac{3x}{3} = x$$
$$g(f(x)) = \cancel{3}\left(\frac{x-11}{\cancel{3}}\right) + 11 = x - 11 + 11 = x$$

Now You Try

(c) $f(x) = \frac{2}{x-7}$ & $\cancel{g(x) = \frac{2}{x} + 7}$

(d) $\cancel{f(x) = 5\sqrt[3]{x} - 7}$ & $g(x) = \frac{(y+7)^3}{125}$

$$g(f(x)) = \sqrt{-2\left(-\frac{1}{2}(x+3)^2 - 4\right) - 8} - 3$$
$$= \sqrt{(x+3)^2 + \cancel{8} - \cancel{8}} - 3$$
$$= x + \cancel{3} - \cancel{3}$$
$$= x$$