Definition of a Limit

$\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

Existence of a Limit

$\lim_{x \to c} f(x) = L$ **if and only if**

$\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

What this says is that if you don't get closer and closer to a number (the "y" or $f(x)$) from both sides of the "x", then there is no **limit** of $f(x)$ at that point.

Now the actual point $f(c)$ may be defined (as in a non-continuous function) at a completely different y (where no limit may occur), but in order for a limit to occur, the x's have to approach a certain y value **from both sides.**

Three Ways to Find a Limit:

1) graphically
2) algebraically
3) numerically (table of values)

Strategies for finding a limit approaching a real number

1. substitute $x = a$
   - if you get a number, that is the limit value
   - if you get $\frac{\#}{0}$, the limit does not exist (the limit might have a direction)
   - if you get $\frac{0}{0}$, do more work and then evaluate the limit
     - factor/reduce
     - find a common denominator
     - simplify complex fractions
     - multiply by the conjugate of an expression with a radical
2. if piecewise: check to see that the right-hand limit = left-hand limit

Example 1 Find each limit.

A. $\lim_{x \to 2}(4x^3) = 4(2)^3 = \boxed{32}$

B. $\lim_{x \to 3} \frac{x-1}{x^2-1} = \frac{3-1}{3^2-1} = \frac{2}{8} = \boxed{\frac{1}{4}}$

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C. $\lim_{x\to 1}\frac{x-1}{x^2-1}$ $\quad \frac{1-1}{1^2-1}=\frac{0}{0}$

$$\lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{1+1}=\boxed{\frac{1}{2}}$$

D. $\lim_{x\to -2}\frac{x^2-2x-8}{x^2-4}$ $\quad \frac{4+4-8}{4-4}=\frac{0}{0}$

$$\lim_{x\to -2}\frac{(x-4)\cancel{(x+2)}}{\cancel{(x+2)}(x-2)}=\lim_{x\to -2}\frac{x-4}{x-2}=\frac{-6}{-4}=\boxed{\frac{3}{2}}$$

E. $\lim_{x\to 0}\frac{(x-2)^2-4}{x}$ $\quad \frac{(-2)^2-4}{0}=\frac{0}{0}$

$$\lim_{x\to 0}\frac{x^{\cancel{2}}-4x\cancel{+4-4}}{\cancel{x}}=\lim_{x\to 0}(x-4)=0-4=\boxed{-4}$$

F. $\lim_{x\to 9}\frac{\sqrt{x}-3}{x-9}$ $\quad \frac{\sqrt{9}-3}{9-9}=\frac{0}{0}$

$$\lim_{x\to 9}\left(\frac{\sqrt{x}-3}{x-9}\right)\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)=\lim_{x\to 9}\frac{\cancel{x+3\sqrt{x}-3\sqrt{x}-9}}{\cancel{(x-9)}(\sqrt{x}+3)}=\lim_{x\to 9}\frac{1}{\sqrt{x}+3}$$

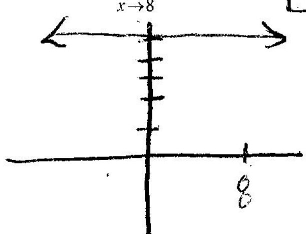
$$\frac{1}{\sqrt{9}+3}=\boxed{\frac{1}{6}}$$

G. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$ $\quad \frac{\sqrt{9}-3}{4-4}=\frac{0}{0}$

$$\lim_{x\to 4}\left(\frac{\sqrt{x+5}-3}{x-4}\right)\left(\frac{\sqrt{x+5}+3}{\sqrt{x+5}+3}\right)=\lim_{x\to 4}\frac{\cancel{x+5+3\sqrt{x+5}-3\sqrt{x+5}-9}}{\cancel{(x-4)}(\sqrt{x+5}+3)}$$

$$=\lim_{x\to 4}\frac{1}{\sqrt{x+5}+3}=\frac{1}{\sqrt{9}+3}=\boxed{\frac{1}{6}}$$

H. $\lim_{x\to 8}5=\boxed{5}$



I. $\lim_{x\to 5}\frac{x+1}{x^2-25}$ $\quad \frac{6}{5^2-25}=\frac{6}{0}$ $\quad \boxed{\text{DNE}}$

One-sided Limits

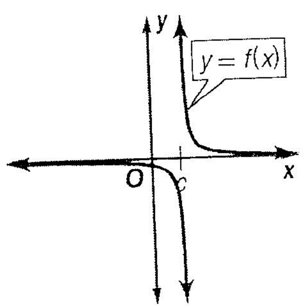
Continuity at a Point
f(x) is continuous at $x = c$, if and only if. $\lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x) = f(c)$.

Continuity on an Open Interval
f(x) is continuous on the interval (a,b), if and only if. f(x) is continuous at all $x \in (a,b)$.

Key Concept **Types of Discontinuity**

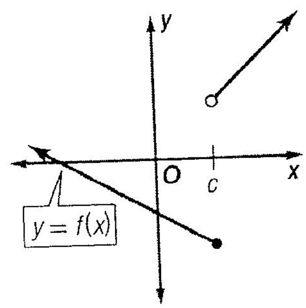
A function has an **infinite discontinuity** at $x = c$ if the function value increases or decreases indefinitely as x approaches c from the left and right.

Example



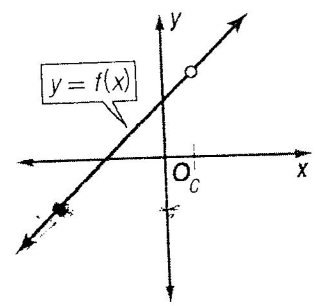
A function has a **jump discontinuity** at $x = c$ if the limits of the function as x approaches c from the left and right exist but have two distinct values.

Example



A function has a **removable discontinuity** if the function is continuous everywhere except for a hole at $x = c$.

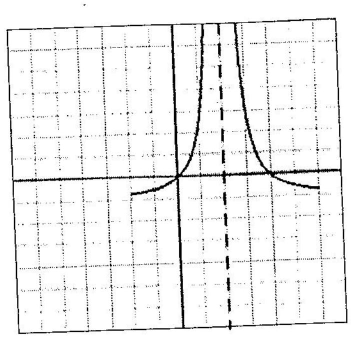
Example



Example 2 **Refer to the graph to find each of the following:**

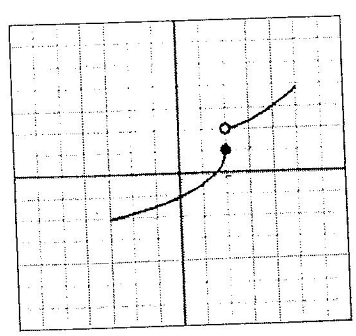
a) the value(s) of x for which the function is discontinuous
b) why it is discontinuous at that value
c) the type of discontinuity
d) whether it is removable (R) or nonremovable (NR) discontinuity

1)



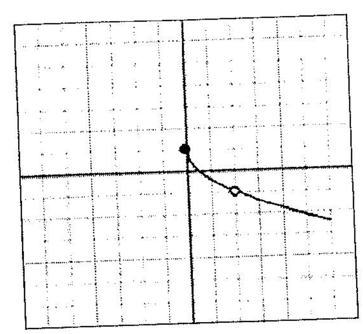
a) $x = 2$
b) $\lim_{x\to 2^-} f(x) = \infty$ $\lim_{x\to 2^+} f(x) = \infty$
c) infinite
d) NR

2)



a) $x = 2.$
b) $\lim_{x\to 2^-} f(x) = 1.$ $\lim_{x\to 2^+} f(x) = 2$
c) jump
d) NR

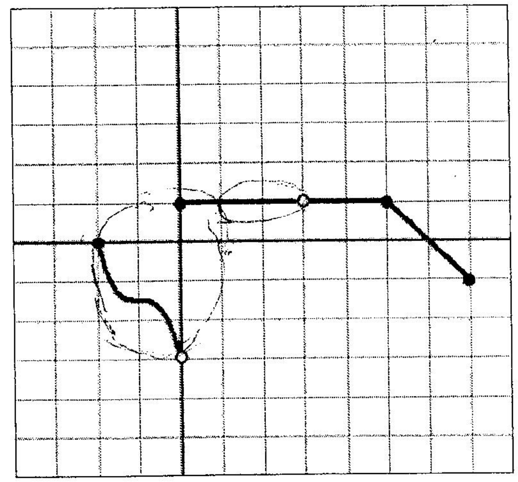
3)



a) $x = 2$
b) f(x) is cont. ever[...] exce[...] x=[...]
c) hole
d) R

Example 3

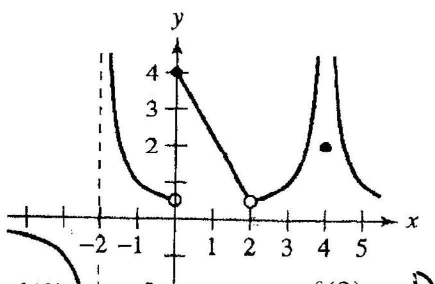
Based on the graph evaluate the following.



1. $\lim_{x\to 0^-} f(x) =$ -3
2. $\lim_{x\to 0^+} f(x) =$ 1
3. $\lim_{x\to 0} f(x) =$ DNE
4. $\lim_{x\to 1^-} f(x) =$ 1
5. $\lim_{x\to 1^+} f(x) =$ 1
6. $\lim_{x\to 1} f(x) =$ 1
7. $\lim_{x\to 5} f(x) =$ 1
8. $f(1) =$ 1
9. $f(0) =$ 1
10. $f(-2) =$ 0
11. $\lim_{x\to 6^-} f(x) =$ 0
12. $\lim_{x\to 6^-} f(x) =$ 0
13. $\lim_{x\to 6} f(x) =$ 0
14. $f(6) =$ 0
15. $\lim_{x\to 3} f(x) =$ 1
16. $f(3) =$ DNE
17. $\lim_{x\to -1} f(x) \approx$ -1.5
18. $f(-1) \approx$ -1.5
19. True or False: $\lim_{x\to c} f(x)$ exists at every c on (1,3)
20. True or False: $\lim_{x\to c} f(x)$ exists at every c on (−2,1)

discont @ $x = 0$

Example 4 Use the graph of f(x) below to find the following:



$f(0) =$ 4

$f(2) =$ DNE

$\lim_{x\to 2^+} f(x) = \frac{1}{2}$

$\lim_{x\to 2^-} f(x) = \frac{1}{2}$

$\lim_{x\to 2} f(x) = \frac{1}{2}$

$\lim_{x\to 0^-} f(x) = \frac{1}{2}$

$\lim_{x\to 0^+} f(x) = 4$

$\lim_{x\to 0} f(x) =$ DNE

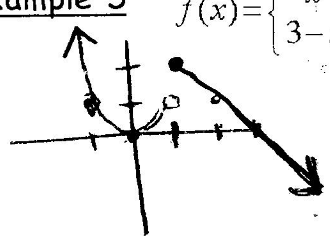
$\lim_{x\to -2^-} f(x) = -\infty$

$\lim_{x\to -2^+} f(x) = \infty$

$\lim_{x\to -2} f(x) =$ DNE

Example 5

$f(x)=\begin{cases} x^2, & x<1 \\ 3-x, & x\ge 1 \end{cases}$ **Find** $\lim_{x\to 1^+} f(x) = 2$ and $\lim_{x\to 1^-} f(x) = 1$



$\lim_{x\to 1} f(x) =$ DNE

Example 6 **Evaluate each limit.**

a. $\lim_{x\to 2^+} \frac{x^2-4}{x-2}$

b. $\lim_{x\to -1^-} \frac{x^2-1}{x^3+1}$

c. $\lim_{x\to 1^+} (2x+3) = \boxed{5}$

$\lim_{x\to 2^+} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}} = 2+2 = \boxed{4}$

$\lim_{x\to -1^-} \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(x^2-x+1)} = \boxed{\frac{-2}{3}}$