AP Calculus AB Name: ____________________

# *Notes ---- Volume: Cross Section Method*

Cross sectional area results from the intersection of a solid with a plane, usually the xy-plane. Volume can be found by breaking up the solids on a given interval into "slices" or cross sections in which the number of cross sections approaches infinity.

Some useful formulas:

Area of a square = $s^2$

Area of a semi-circle = $\frac{1}{2}\pi r^2$

Area of a triangle = $\frac{1}{2}bh$

Area of an equilateral triangle = $\frac{\sqrt{3}}{4}s^2$

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**<u>Volume with Known Cross Sections</u>**

- If we know the formula for the area of the cross section, we can find the volume of the solid having this cross section with the help of the definite integral.

If the cross section is **<u>perpendicular to the x-axis</u>** and its area is a function of $x$, say $A(x)$, then the volume of the solid on $[a, b]$ is given by

$$V = \int_a^b A(x)dx$$

If the cross section is **<u>perpendicular to the y-axis</u>** and its area is a function of $y$, say $A(y)$, then the volume of the solid on $[a, b]$ is given by

$$V = \int_a^b A(y)dy$$

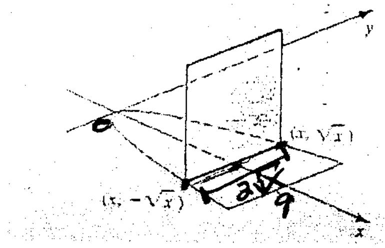
- The solid is NOT being revolved around an axis. It is placed perpendicular along an axis.
- In order to calculate the volume you need to identify 3 pieces of information:
  1. **Base Region** The base region graph shows the length of a side of the cross section shape and the limits of integration, $[a, b]$.
  2. **Cross Sections Shape** The shape will determine the area formula, $A(x)$ or $A(y)$, to use in the integrand.
  3. **Perpendicular Axis** $\perp$ to the $x-axis \rightarrow dx$

     $\perp$ to the $y-axis \rightarrow dy$

Ex 1) Let $R$ be the region bounded by the graphs of $x = y^2$ and $x = 9$. Find the volume of the solid that has $R$ as its base if every cross section by a plane perpendicular to the x-axis has the given shape.

use $dx$

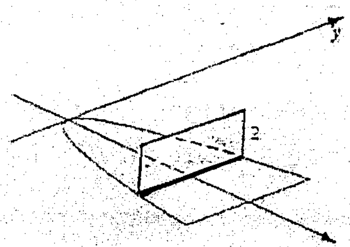
a) A square




$$V = \int_0^9 4x\,dx = 2x^2 + C \Big|_0^9$$
$$= 2(9)^2 - 2(0)^2 = \boxed{162}$$

$$A = s^2 = (2\sqrt{x})^2 = 4x$$

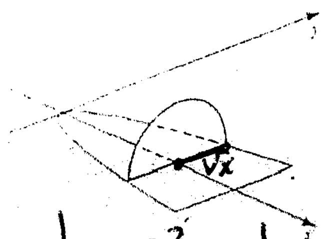
b) A rectangle with height 2




$$V = \int_0^9 4\sqrt{x}\,dx = \frac{8}{3}x^{\frac{3}{2}} + C \Big|_0^9$$
$$= \frac{8}{3}(9)^{\frac{3}{2}} - \frac{8}{3}(0)^{3/2} = \boxed{72}$$

$$A = bh = 2\sqrt{x} \cdot 2 = 4\sqrt{x}$$

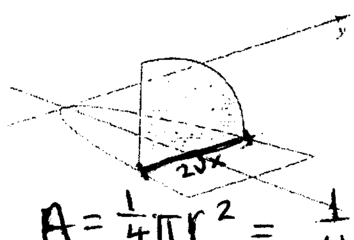
c) A semicircle




$$V = \int_0^9 \frac{1}{2}\pi x\,dx = \frac{1}{4}\pi x^2 + C \Big|_0^9$$
$$\boxed{\frac{81}{4}\pi}$$

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(\sqrt{x})^2 = \frac{1}{2}\pi x$$

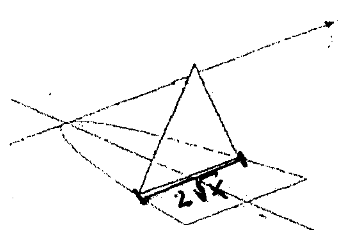
d) A quartercircle




$$V = \int_0^9 \pi x\,dx = \frac{1}{2}\pi x^2 + C \Big|_0^9$$
$$\boxed{\frac{81\pi}{2}}$$

$$A = \frac{1}{4}\pi r^2 = \frac{1}{4}\pi(2\sqrt{x})^2 = \pi x$$

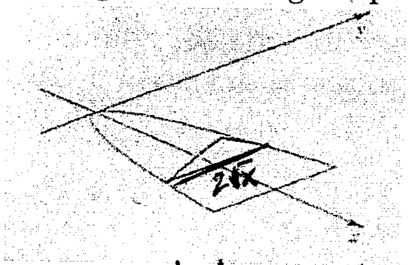
e) An equilateral triangle




$$V = \int_0^9 \sqrt{3}\,x\,dx = \frac{\sqrt{3}}{2}x^2 + C \Big|_0^9$$
$$\boxed{\frac{81\sqrt{3}}{2}}$$

$$A = \frac{\sqrt{3}}{4}s^2 = \frac{\sqrt{3}}{4}(2\sqrt{x})^2 = \sqrt{3}x$$

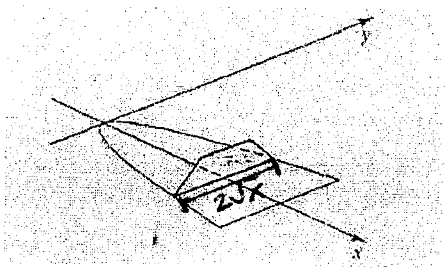
f) A triangle with a height equal to $\frac{1}{4}$ of the length of the base




$$V = \int_0^9 \frac{1}{2}x\,dx = \frac{1}{4}x^2 + C\Big|_0^9 = \boxed{\frac{81}{4}}$$

$$A = \frac{1}{2}bh = \frac{1}{2}\cdot 2\sqrt{x}\cdot\frac{1}{4}\cdot 2\sqrt{x} = \frac{1}{2}x$$

g) A trapezoid with lower base in the xy-plane, upper base equal to $\frac{1}{2}$ the length of the lower base, and height equal to $\frac{1}{4}$ the length of the lower base.

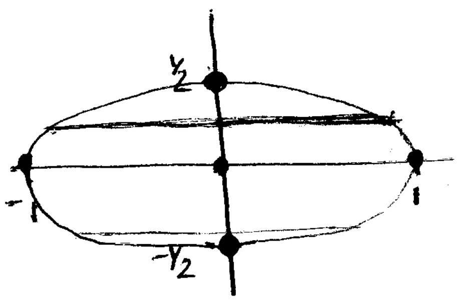



$$V = \int_0^9 \frac{3}{4}x\,dx = \frac{3}{8}x^2 + C\Big|_0^9 = \boxed{\frac{243}{8}}$$

$$A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}\cdot\frac{1}{4}\cdot 2\sqrt{x}\left(2\sqrt{x} + \frac{1}{2}\cdot 2\sqrt{x}\right) = \frac{1}{4}\sqrt{x}\left(3\sqrt{x}\right) = \frac{3}{4}x$$

Ex 2) Suppose you have a solid with an elliptical base whose equation is $x^2 + 4y^2 = 1$. Semicircular cross-sectional slices are taken perpendicular to the y-axis. Find the volume of the solid.

use $dy$




$$x^2 = 1 - 4y^2$$

$$x = \pm\sqrt{1 - 4y^2}$$

$$\frac{x^2}{1} + \frac{y^2}{\frac{1}{4}} = 1$$

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\sqrt{1 - 4y^2}\right)^2 = \frac{1}{2}\pi\left(1 - 4y^2\right)$$

$$V = \int_{-1/2}^{1/2} \frac{1}{2}\pi\left(1 - 4y^2\right)dy \quad \text{OR} \quad 2\int_0^{1/2} \frac{1}{2}\pi\left(1 - 4y^2\right)dy$$

$$\pi\left(y - \frac{4}{3}y^3 + C\right)\Big|_0^{1/2}$$

$$\pi\left(\frac{1}{2} - \frac{1}{6}\right)$$

$$\boxed{\frac{\pi}{3}}$$

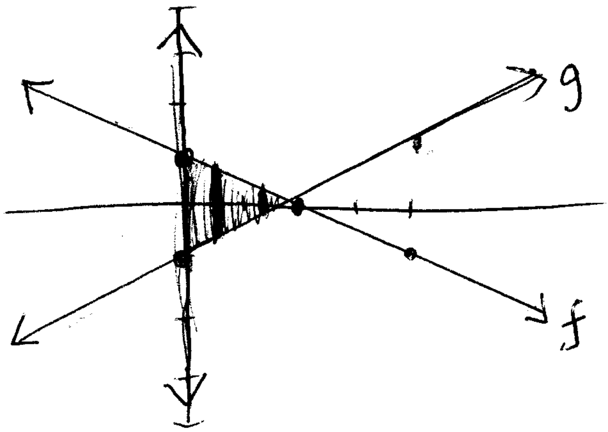
Ex 3) Find the volume of the solid whose base is bounded by the lines $f(x) = 1 - \frac{x}{2}$, $g(x) = -1 + \frac{x}{2}$, and $x = 0$. The cross-sections perpendicular to the x-axis are equilateral triangles.

use $dx$

$$A = \frac{\sqrt{3}}{4} s^2$$

$$= \frac{\sqrt{3}}{4}\left(1 - \frac{x}{2} - \left(-1 + \frac{x}{2}\right)\right)^2$$

$$= \frac{\sqrt{3}}{4}(2 - x)^2$$




$$V = \int_0^2 \frac{\sqrt{3}}{4}(2-x)^2\,dx$$

$$\frac{\sqrt{3}}{4} \cdot \frac{8}{3} = \boxed{\frac{2\sqrt{3}}{3}}$$