Example 6 Who is the Plurality with Elimination winner?

Preference	15 voters	7 voters	13 voters	5 voters	2 voters
1st	Helen	~~Eddie~~	Grover	~~Donna~~	~~Eddie~~
2nd	~~Donna~~	~~Donna~~	~~Flora~~	Grover	~~Flora~~
3rd	~~Eddie~~	Grover	~~Eddie~~	~~Flora~~	Grover
4th	~~Flora~~	~~Flora~~	~~Donna~~	~~Eddie~~	~~Donna~~
5th	Grover	Helen	Helen	Helen	Helen

42 votes
majority
7 21

Round 1		Round 2		Round 3		Round 4	
D 5		D 5		E 9	elim E	G 27	$\boxed{\text{Grover wins}}$
E 9	elim. F	E 9	elim D	G 18		H 15	
F 0		G 13		H 15			
G 13		H 15					
H 15							

Borda Count

- if there are m candidates, then for each vote, m points are assigned to the 1st choice, (m – 1) points are assigned to the 2nd choice, and so on
- the candidate that receives more points in total than any other is declared the winner
- invented in 1770 by the French mathematician and physicist Jean-Charles Borda

Advantages

- Incorporates all information from a preference ballot
- takes candidate with best average ranking
- preferable when comparing a large number of candidates

Example 7 The Math Appreciation Society is electing its president. The candidates are Alisha (A), Boris (B), Carmen (C), and Dave (D). Each of the 37 members votes with a preference ballot. Who should be the winner using a Borda Count?

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

A: $4(14)+3(0)+2(0)+1(23)=79$

B: $4(4)+3(24)+2(9)+1(0)=106$ $\boxed{\text{B wins}}$

C: $4(11)+3(8)+2(18)+1(0)=104$

D: $4(8)+3(5)+2(10)+1(14)=81$

Example 8 Determine the Borda count winner.

Preference	12 voters	9 voters	3 voters	8 voters
1st	Charles	Bonnie	*Charles*	Adam
2nd	Adam	Charles	*Adam*	Bonnie
3rd	Bonnie	Adam	Bonnie	Charles

A: $3(8)+2(15)+1(9)=63$

B: $3(9)+2(8)+1(15)=58$

C: $3(15)+2(9)+1(8)=71$

Charles wins

Run-off Method

- Voters cast a preference ballot.
- Eliminate all candidates except the two (2) with the most 1st choice votes.
- Reassign votes from eliminated candidates by giving to the remaining candidate ranked higher.

Example 9 The Math Appreciation Society is electing its president. The candidates are Alisha (A), Boris (B), Carmen (C), and Dave (D). Each of the 37 members votes with a preference ballot. Who should be the winner using the Run-off method?

Number of Voters	14	10	8	4	1
1st choice	A	C	~~D~~	~~B~~	C
2nd choice	~~B~~	~~B~~	C	~~D~~	~~D~~
3rd choice	C	~~D~~	~~B~~	C	~~B~~
4th choice	~~D~~	A	A	A	A

A 14 ←
B 4
C 11 ←
D 8

A 14
C 23

C wins

Example 10 Determine a winner using the Run-off method.

Number of voters	49	48	3
1st choice *4pts*	A	B	~~C~~
2nd choice *3pts*	B	~~D~~	B
3rd choice *2pts*	~~C~~	~~C~~	~~D~~
4th choice *1pt*	~~D~~	A	A

A 49
B 48
C 3
D 0

A 49
B 51

B wins

Example 11 Ms. Powell is rewarding her ICM students by giving them an ice cream party. She takes a poll of favorite ice cream flavors among her students where A = vanilla, B = chocolate, C = strawberry, and D = mint chocolate chip (her favorite). Who is the run-off method winner?

Number of voters	27	19	8	15	2
1st choice	*B*	*A*	~~*D*~~	~~*C*~~	*A*
2nd choice	~~*D*~~	~~*D*~~	~~*C*~~	*A*	~~*C*~~
3rd choice	*A*	~~*C*~~	*A*	~~*D*~~	~~*D*~~
4th choice	~~*C*~~	*B*	*B*	*B*	*B*

A 21
B 27
C 15
D 8

A 44
B 27

Vanilla wins

Approval Method

- If using an "approval ballot", count total # of votes for each candidate.
- If using a preference ballot, usually the top 2 choices are considered to be approval votes.
- It is NOT a group ranking method

Example 12 (approval ballot) A group of friends is trying to decide upon a movie to watch. 3 choices are provided, and each person is asked to mark with an "X" which movies they are willing to watch. Find the winner using the Approval voting method.

	Bob	Ann	Marv		Eve	Omar	Lupe	Dave	Tish	Jim
Titanic		X	X			X		X		X
Scream	X		X	X		X	X		X	
The Matrix	X	X	X	X	X		X			X

T: 5 S: 6 M: 7 $\boxed{\text{The Matrix wins}}$

Example 13 (preference ballot) Determine the winner using the Approval method (consider the top 2 choices as approval votes).

	80	15	5
1st choice	A	B	C
2nd choice	B	C	B
~~3rd choice~~	~~C~~	~~A~~	~~A~~

A 80
B 80+15+5 = 100
C 15+5 = 20

$\boxed{\text{B wins}}$

The Condorcet Method

- Different methods of determining group rankings often give different results.
- Marquis de Condorcet proposed that any choice that can beat each of the others in a head to head race should win.

Example 14

A	B	C	D
B	C	B	B
C	D	D	C
D	A	A	A
8	5	6	7

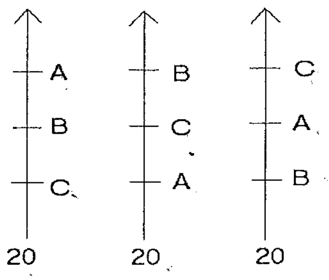
Total # of voters: 26

A vs B A vs C A vs D
8 18
B vs C B vs D C vs D
20 6 19 7

Determine the winner using the Condorcet method. $\boxed{\text{B wins}}$

Example 15

The Condorcet Method seems hard to argue with, but it does not always produce a winner.



- A is preferred to B by a majority of voters. A vs B 40 20
- B is preferred to C by a majority of voters. B vs C 40 20
- So we might think that A is preferred to C by a majority of voters. A vs C 20 40
- However, C beats A, so this is not true.

Group ranking methods may violate the transitive property

This seems contrary to intuition and is known as a paradox.

Pairwise Comparison Method

- Compare each candidate one-on-one.
- A win is worth one point, a tie worth $\frac{1}{2}$ point, and a loss worth nothing.
- Do all possible combinations.
- It may help to make a chart like this: A | B

Example 16 The Math Appreciation Society is electing its president. The candidates are Alisha (A), Boris (B), Carmen (C), and Dave (D). Each of the 37 members votes with a preference ballot. Who should be the winner using the Pairwise Comparison method?

A | B 14 | 23 A | C 14 | 23 A | D 14 | 23

B | C 18 | 19 B | D 28 | 9 C | D 25 | 12

A 0
B 2
C 3
D 1

Number of Voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

$\boxed{\text{C wins}}$

Example 17 The Los Angeles LAXers are getting number one choice in upcoming draft. The list is narrowed down to Allen, Byers, Castillo, Dixon and Evans. Who wins with Pairwise Comparison?

A | B 7 | 15 A | C 16 | 6 A | D 13 | 9 A | E 18 | 4

B | C 10 | 12 tie B | D 11 | 11 B | E 14 | 8

C | D 12 | 10 C | E 10 | 12 D | E 18 | 4

A 3
B 2½
C 2
D 1½
E 1

Preference Schedule: Laxers Draft

Number of voters	2	6	4	1	1	4	4
First choice	A	B	B	C	C	D	E
Second choice	D	A	A	B	D	A	C
Third choice	C	C	D	A	A	E	D
Fourth choice	B	D	E	D	B	C	B
Fifth choice	E	E	C	E	E	B	A

22 voters

$\boxed{\text{A wins}}$

Arrow's Conditions

Ten representatives of the language clubs at HSHS are meeting to select a location for the clubs' annual dinner. They must choose between a Chinese, French, Italian, or Mexican restaurant.

M	F	I
I	C	F
F	M	C
C	I	M
4	3	3

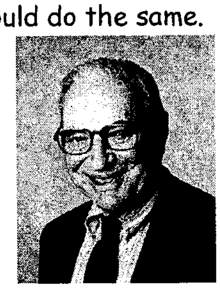
- Kyle suggests that because the last 2 dinners have been held at Mexican and Chinese restaurants, this year's dinner should be at either an Italian or French restaurant. They vote 7 to 3 in favor of the Italian restaurant.

- John doesn't like Italian food and says that the new Mexican restaurant is really good. He proposes that the group choose between Italian and Mexican. They vote 7 to 3 to hold the dinner at the Mexican restaurant.

- Kaitlin's parents own a Chinese restaurant and say that she can get a group discount. The group votes between the Mexican and Chinese restaurant and selects the Chinese restaurant by a 6 to 4 margin.

* If we look back at their original preferences, we see that French food was preferred to Chinese food in every case, yet they voted for Chinese food.

* **Kenneth Arrow**, a mathematician, said this is not fair. He believed that if every member of a group prefers one choice to another, then the group ranking should do the same. (He thought pairwise voting is unfair.)



Arrow's Conditions

1. Nondictatorship - The preferences of a single individual should not become the group ranking without considering the preferences of others.

2. Individual Sovreignty - Each individual should be allowed to order the choices in any way and to indicate ties.

3. Unanimity - If every individual prefers one choice to another, then the group ranking should do the same.

4. Freedom from Irrelevant Alternatives - The winning choice should still win if one of the other choices is removed.

5. Uniqueness of Group Ranking - The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive.

❖ Arrow proved that no method, known or unknown, could always obey all 5 conditions. Any group-ranking method will violate at least one of Arrow's conditions in the same situation. Although a perfect group ranking will never be found, current methods can still be improved.