

Notes: Inverse Trig Functions, Derivatives & Integration

Six basic trig functions

Three basic trig functions:

$$\sin \theta = \frac{\text{Opp}}{\text{Hyp}}$$

$$\cos \theta = \frac{\text{Adj}}{\text{Hyp}}$$

$$\tan \theta = \frac{\text{Opp}}{\text{Adj}}$$

Reciprocal trig functions:

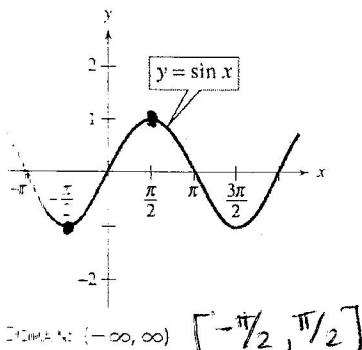
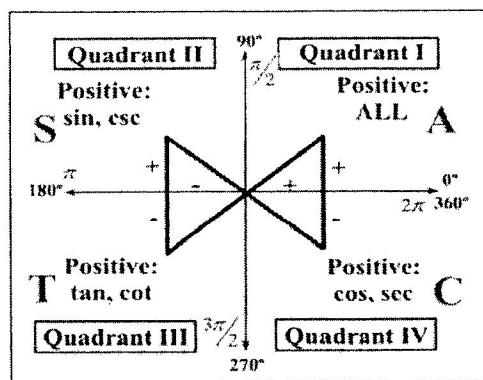
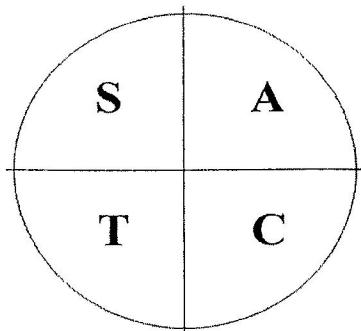
$$\csc \theta = \frac{\text{Hyp}}{\text{Opp}}$$

$$\sec \theta = \frac{\text{Hyp}}{\text{Adj}}$$

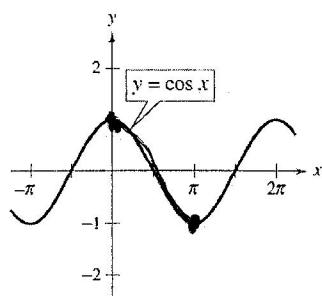
$$\cot \theta = \frac{\text{Adj}}{\text{Opp}}$$



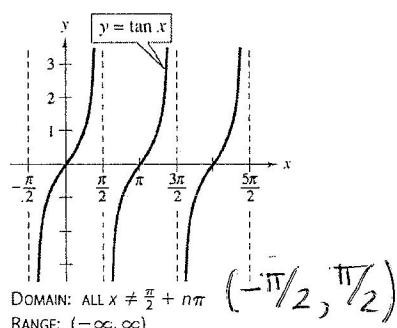
(sine: opp./hyp.; cosine: adj./hyp.; tangent: opp./adj.)



DOMAIN: $(-\infty, \infty)$ $[-\frac{\pi}{2}, \frac{\pi}{2}]$
RANGE: $[-1, 1]$



DOMAIN: $(-\infty, \infty)$ $[0, \pi]$
RANGE: $[-1, 1]$



DOMAIN: ALL $x \neq \frac{\pi}{2} + n\pi$ $(-\frac{\pi}{2}, \frac{\pi}{2})$
RANGE: $(-\infty, \infty)$

Function	Domain	Range
$\arcsin x$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\arccos x$	$[-1, 1]$	$[0, \pi]$
$\arctan x$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
$\text{arc csc } x$	$(-\infty, -1] \cup [1, \infty)$	$[-\frac{\pi}{2}, \frac{\pi}{2}] y \neq 0$
$\text{arc sec } x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] y \neq \frac{\pi}{2}$
$\text{arc cot } x$	$(-\infty, \infty)$	$(0, \pi)$

Example 1 Evaluate:

a. $\sin^{-1}(0) = 0$

b. $\tan^{-1}(1) = \frac{\pi}{4}$

c. $\tan^{-1}(-1) = -\frac{\pi}{4}$

d. $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

e. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$

f. $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

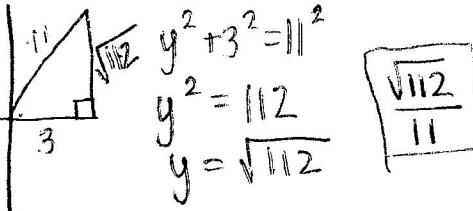
g. $\operatorname{arccot}(1) = \frac{\pi}{4}$

h. $\arccos(-0.628)$ [use a calculator]

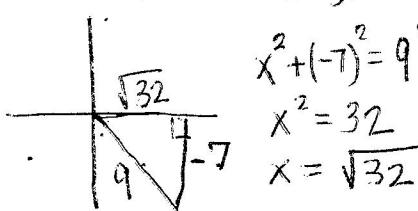
2.2497

Example 2 Evaluate:

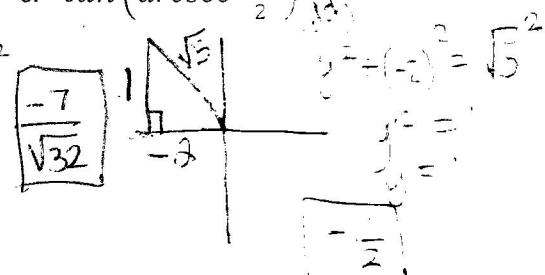
a. $\sin(\arccos \frac{3}{11})$ adj hyp



b. $\tan(\arcsin -\frac{7}{9})$ opp hyp



c. $\tan(\operatorname{arcsec} \frac{-\sqrt{5}}{2})$ opp hyp



Derivatives of Inverse Trig Functions

Let u be a differentiable function of x .

$$\frac{d}{dx} [\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} [\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx} [\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx} [\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx} [\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx} [\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Example 3 Find each derivative.

a. $y = \sin^{-1}(2x)$

$$y' = \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 = \frac{2}{\sqrt{1-4x^2}}$$

b. $y = \tan^{-1}(3x)$

$$y' = \frac{1}{1+(3x)^2} \cdot 3 = \frac{3}{1+9x^2}$$

c. $y = \sec^{-1}(e^{2x})$

$$y' = \frac{1}{|e^{2x}| \sqrt{(e^{2x})^2 - 1}} \cdot e^{2x} \cdot 2 = \frac{2}{\sqrt{e^{4x}-1}}$$

d. $g(x) = \frac{\arcsin(3x)}{x}$

$$g'(x) = x \cdot \frac{1}{\sqrt{1-(3x)^2}} \cdot 3 - \frac{\arcsin(3x) \cdot 1}{x^2}$$

Integration Involving Inverse Trig Functions

Let u be a differentiable function of x , and let $a > 0$.

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + C$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C$$

Integrate.

$$\begin{aligned} a. \int \frac{2}{9+4x^2} dx \\ u^2 = 9 \quad u^2 = 4x^2 \\ a = 3 \quad u = 2x \\ \frac{du}{dx} = 2 \\ du = 2dx \end{aligned}$$

$$\begin{aligned} \int \frac{du}{a^2 + u^2} \\ \frac{1}{a} \arctan \frac{u}{a} + C \\ \boxed{\frac{1}{3} \arctan \frac{2x}{3} + C} \end{aligned}$$

$$\begin{aligned} b. \int \frac{3x}{\sqrt{1-9x^2}} dx \\ u = 1-9x^2 \quad \frac{-1}{18} \cdot 3 \int \frac{du}{\sqrt{u}} \\ \frac{du}{dx} = -18x \quad -\frac{1}{6} \int u^{-\frac{1}{2}} du \\ -\frac{1}{18} du = xdx \\ -\frac{1}{3} u^{\frac{1}{2}} + C \\ \boxed{-\frac{1}{3} \sqrt{1-9x^2} + C} \end{aligned}$$

$$\begin{aligned} c. \int \frac{x}{2x^2\sqrt{4x^4-36}} dx \\ u^2 = 4x^4 \quad a^2 = 36 \\ u = 2x^2 \quad a = 6 \\ \frac{du}{dx} = 4x \\ \frac{1}{4} du = xdx \\ e. \int \frac{4}{\sqrt{3} \cdot 9+x^2} dx \end{aligned}$$

$$\begin{aligned} \frac{1}{4} \int \frac{du}{u\sqrt{u^2-a^2}} \\ \frac{1}{4} \cdot \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + C \\ \frac{1}{24} \operatorname{arcsec} \frac{|2x^2|}{6} + C \\ \boxed{\frac{1}{24} \operatorname{arcsec} \frac{x^2}{3} + C} \end{aligned}$$

$$\begin{aligned} d. \int_0^1 \frac{1}{\sqrt{4-x^2}} dx \\ a^2 = 4 \quad u^2 = x^2 \\ a = 2 \quad u = x \\ \frac{du}{dx} = 1 \\ du = dx \\ \int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C \\ \arcsin \frac{x}{2} + C \Big|_0^1 \\ \arcsin \frac{1}{2} + C - (\arcsin 0) + C \end{aligned}$$

$$f. \int \frac{x^4-1}{x^2+1} dx$$

$$\int \frac{(x^2+1)(x^2-1)}{x^2+1} dx = \frac{1}{3} x^3 - x + C$$

$$\begin{aligned} 4 \int \frac{du}{a^2+u^2} &= 4 \cdot \frac{1}{a} \arctan \frac{u}{a} + C \\ &= \frac{4}{3} \arctan \frac{x}{3} + C \Big|_{\sqrt{3}}^3 \\ &= \frac{4}{3} \arctan 1 + C - \left(\frac{4}{3} \arctan \frac{\sqrt{3}}{3} + C \right) \\ &= \frac{4}{3} \cdot \frac{\pi}{4} - \frac{4}{3} \cdot \frac{\pi}{6} = \frac{\pi}{3} - \frac{2\pi}{9} = \frac{\pi}{9} \end{aligned}$$

$$g. \int \frac{t}{t^4+16} dt$$

$$\text{complete the square}$$

$$\begin{aligned} u^2 = t^4 \quad a^2 = 16 \\ u = t^2 \quad a = 4 \\ \frac{du}{dt} = 2t \\ \frac{1}{2} du = tdt \\ \frac{1}{2} \int \frac{du}{u^2+a^2} \end{aligned}$$

$$\frac{1}{2} \cdot \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\boxed{\frac{1}{8} \arctan \frac{t^2}{4} + C}$$

$$\begin{aligned} h. \int \frac{dx}{x^2+4x+13} &\text{ complete the square} \\ &\frac{x^2+4x+4+13-4}{(x+2)^2+9} \\ &\frac{(x+2)^2+9}{(x+2)^2+9} \\ &\frac{dx}{(x+2)^2+9} \\ &\frac{du}{u^2+a^2} \\ &\frac{1}{a} \arctan \frac{u}{a} + C \\ &\frac{1}{3} \arctan \frac{x+2}{3} + C \end{aligned}$$