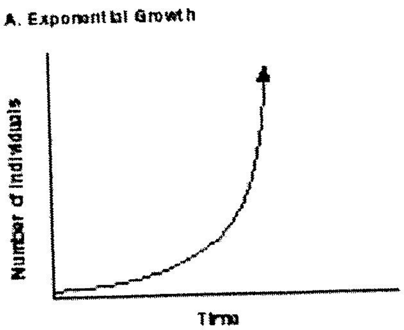
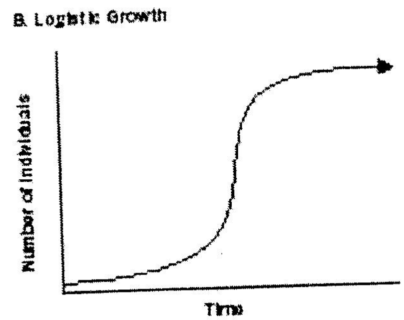
Calculus AB

Notes---- Logistic Differential Equations

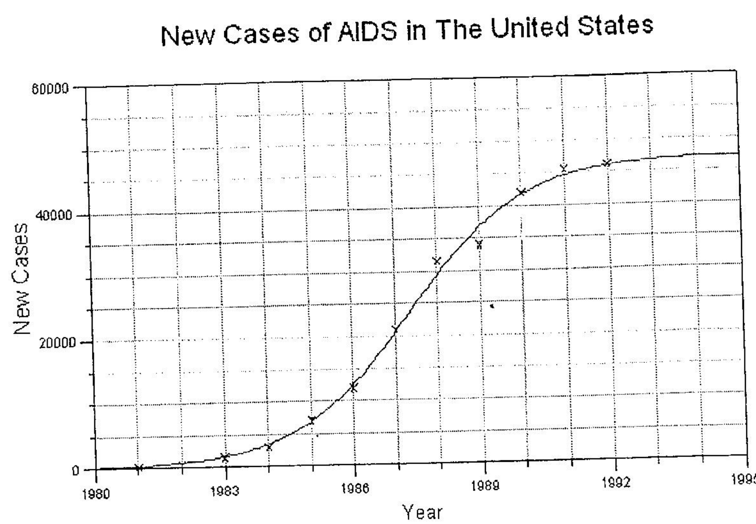
The exponential is only bounded below. However, for population growth there exists some upper limit past which growth cannot occur.





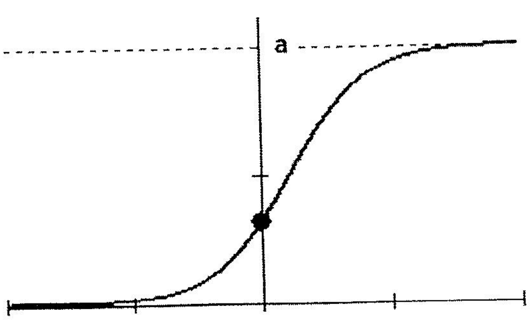
A logistic differential equation has the form: $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

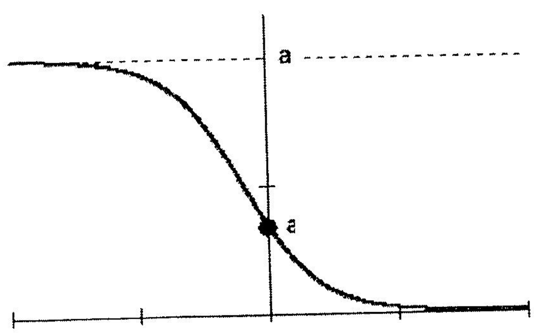
Where k and L are positive constants. L is the carrying capacity or the limit to growth which can be sustained or supported as time t increases.

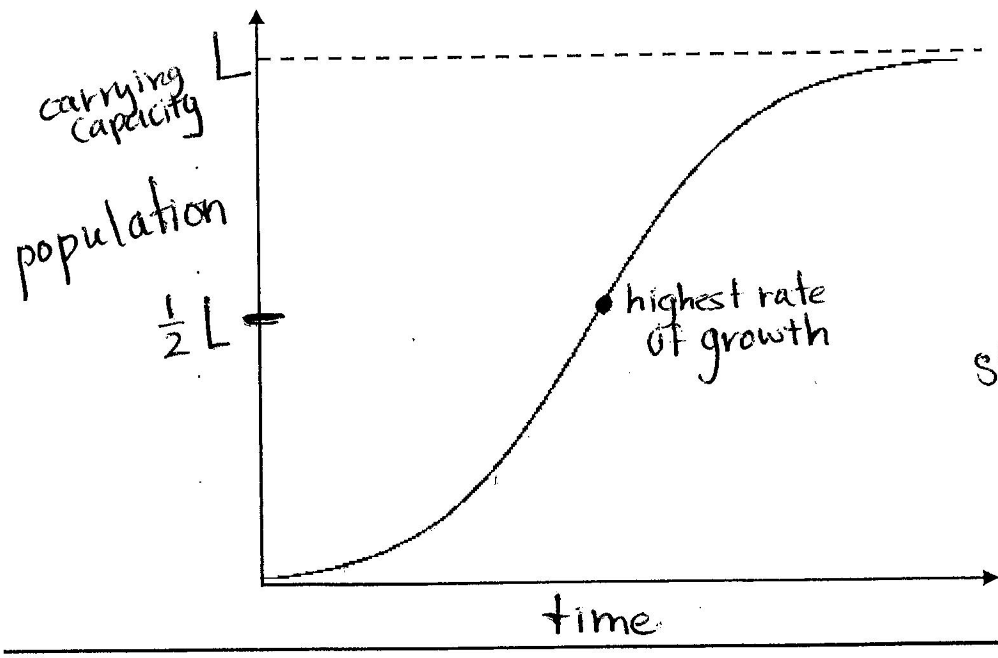


Note: If y is between 0 and the carrying capacity L, then $\frac{dy}{dt} > 0$ and the population increas

If $y > L$ the $\frac{dy}{dt} < 0$ and the population decreases.







Slope of the curve = rate of growth of the pop.

Finding the solution of the logistic equation.

Ex1) $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$

$$\underbrace{\int \frac{dy}{y(1-\frac{y}{L})}}_{\text{partial fractions}} = \int k\,dt$$

$$-\ln|L-y| + \ln|y| = kt + C$$

$$\ln|L-y| - \ln|y| = -kt + C$$

$$\ln\left|\frac{L-y}{y}\right| = -kt + C \qquad e^{-kt+C} \to e^{-kt} \cdot \overbrace{e^{C}}^{b}$$

$$\frac{L-y}{y} = be^{-kt}$$

$$\frac{L}{y} - 1 = be^{-kt}$$

$$\frac{L}{y} = 1 + be^{-kt}$$

$$\frac{y}{L} = \frac{1}{1+be^{-kt}}$$

$$y = \frac{L}{1+be^{-kt}}$$

$(0, 40)$ $(5, 104)$

Ex 2) A state game commission releases 40 elk into a game refuge. After 5 years, the e... population is 104. The commission believes that the environment can support no m... than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right) \text{ for } 40 \leq p \leq 4000$$

Where t is the number of years.

a) Write a model for the elk population in terms of t.

$$p(t) = \frac{L}{1+be^{-kt}} = \frac{4000}{1+be^{-kt}}$$

$$40 = \frac{4000}{1+be^{-k(0)}}$$

$$40 = \frac{4000}{1+b}$$

$$40(1+b) = 4000$$

$$1+b = 100$$
$$b = 99$$

$$p(t) = \frac{4000}{1+99e^{-kt}}$$

$$104 = \frac{4000}{1+99e^{-5k}}$$

$$k = .19436102$$

$$p(t) = \frac{4000}{1+99e^{-.194t}}$$

b) Use the model to estimate the elk population after 15 years.

$$p(15) = 628.538$$

$$\boxed{628 \text{ elk}}$$

c) Find the limit of the model as $t \to \infty$.

$$\lim_{t \to \infty} p(t) = 4000$$

Ex 3) Suppose the population of the bears in a national park grows according to the logistic differential equation. $\frac{dP}{dt} = 5P - 0.002P^2$, where P is the number of bears in time t years

a) If $P(0) = 100$, then the $\lim_{t\to\infty} P(t) =$ 2500

$$\frac{dp}{dt} = 5p\left(1 - \frac{P}{2500}\right)$$

b) If $P(0) = 1500$, then the $\lim_{t\to\infty} P(t) =$ 2500

$k = 5$

$L = 2500$

c) If $P(0) = 3000$, then the $\lim_{t\to\infty} P(t) =$ 2500

d) How many bears are in the park when the population is growing the fastest?

$$\frac{1}{2}(2500) = \boxed{1250 \text{ bears}}$$

Ex 4) Suppose a population of wolves grows to the logistic differential equation $\frac{dP}{dt} = 3P - 0.01P^2$ where P is the number of wolves at time t years. Which of the following statements are true?

$$\frac{dp}{dt} = 3P\left(1 - \frac{P}{300}\right)$$

$k = 3$

$L = 300$

I. $\lim_{t\to\infty} P(t) = 300$ ✓

II. The growth rate of the wolf population is greatest at $P = 150$. ✓

III. If $P > 300$, the population of wolves is increasing ✗

a) I only
b) II only
c) I and II
d) II and III
e) All of them.

Ex 5) A population of animals growth is modeled by a function P that satisfies the logistic differential equation $\frac{dP}{dt} = 0.01P(100 - P)$, where time t is measured in years.

a) If $P(0) = 20$, solve for P as a function of t.
b) Using the answer in part a), find P when $t = 3$ years.
c) Using the answer in part a), find t when $P = 80$ animals.

$$\frac{dp}{dt} = 100(0.01)P\left(1 - \frac{P}{100}\right)$$

$$\frac{dp}{dt} = 1P\left(1 - \frac{P}{100}\right)$$

$k = 1$

$L = 100$

a) $p(t) = \frac{L}{1+be^{-kt}} = \frac{100}{1+be^{-1t}}$

$20 = \frac{100}{1+be^{-1(0)}}$

$20 = \frac{100}{1+b}$

$20(1+b) = 100$

$1+b = 5$

$b = 4$

$\boxed{p(t) = \frac{100}{1+4e^{-t}}}$

$b = \frac{L}{P_0} - 1$

b) $p(3) = 83.393$

$\boxed{83 \text{ animals}}$

c) $80 = \frac{100}{1+4e^{-t}}$

$\boxed{t = 2.773 \text{ yrs}}$