

Optimization notes Day 1

Example 1 Find two positive numbers whose product is 192, and the sum of the first plus three times the second is a minimum.

Let $x = 1^{st} \#$
 $y = 2^{nd} \#$

minimize $S = x + 3y$

$xy = 192$

$y = \frac{192}{x}$

$S = x + 3\left(\frac{192}{x}\right) = x + \frac{576}{x} = x + 576x^{-1}$

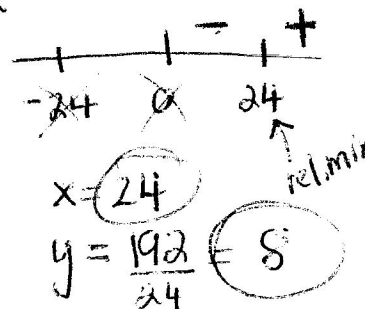
$S' = 1 - 576x^{-2} = 1 - \frac{576}{x^2}$

S' undef at $x=0$

$x^2 \left(1 - \frac{576}{x^2}\right) = 0 \cdot x^2$

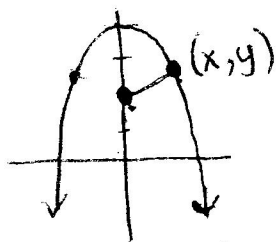
$x^2 - 576 = 0$

$x^2 = 576 \quad x = \pm 24$



$x = 24$
 $y = \frac{192}{24} = 8$

Example 2 Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$?



minimize the distance between $(0, 2)$ and (x, y)

$d = \sqrt{(x-0)^2 + (y-2)^2}$

$d = \sqrt{x^2 + (4 - x^2 - 2)^2}$

$d = \sqrt{x^2 + 4 - 4x^2 + x^4}$

$d = \sqrt{x^4 - 3x^2 + 4}$

$d' = \frac{1}{2}(x^4 - 3x^2 + 4)^{-1/2} (4x^3 - 6x)$

$d' = \frac{4x^3 - 6x}{2\sqrt{x^4 - 3x^2 + 4}}$

$4x^3 - 6x = 0$

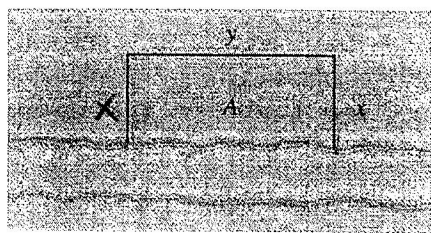
$2x(2x^2 - 3) = 0$

$2x = 0 \quad 2x^2 - 3 = 0$
 $x = 0 \quad x = \pm \sqrt{\frac{3}{2}}$

$x = 0$ is a rel. min.

$y = 4 - \left(-\sqrt{\frac{3}{2}}\right)^2 = 2.5$
 $y = 4 - \left(\sqrt{\frac{3}{2}}\right)^2 = 2.5$
 $\left(-\sqrt{\frac{3}{2}}, 2.5\right)$ and $\left(\sqrt{\frac{3}{2}}, 2.5\right)$

Example 3 A farmer wants to fence in a rectangular region. A river runs along one of the sides of the region, so the farmer only needs to put the fence on three of the sides. If the farmer has 200 yards of fencing to use, find the largest possible area which can be enclosed by the fence. What are the dimensions of the region?



maximize $A = xy$

$2x + y = 200$

$A = x(-2x + 200) = -2x^2 + 200x$

$y = -2x + 200$

$y = -2(50) + 200 = 100$

$A' = -4x + 200$

$-4x + 200 = 0$

$x = 50$

sign chart for A' :
 $+$ $-$
 0 50

rel max

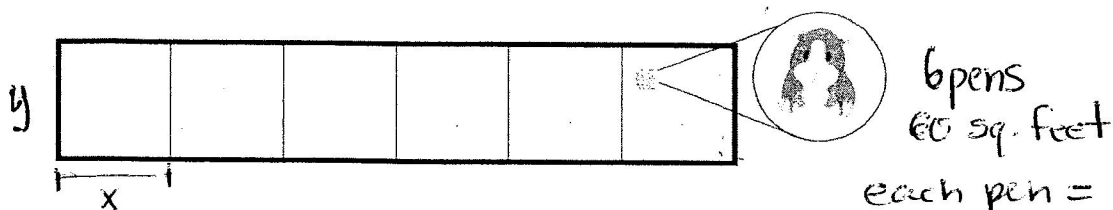
dimensions

50 yd by 100 yd

max area

$50(100) = 5000 \text{ yd}^2$

Example 4 Your dream of becoming a hamster breeder has finally come true. You are constructing a set of rectangular pens in which to breed your furry friends. The overall area you are working with is 60 square feet, and you want to divide the area up into six pens of equal size as shown below.



The cost of the outside fencing is \$10 a foot. The inside fencing costs \$5 a foot. You wish to minimize the cost of the fencing. Find the exact dimensions of each pen that will minimize the cost of the breeding ground. What is the total cost?

$$C = 10(12x + 2y) + 5(5y) = 120x + 120y + 25y = 120x + 145y$$

$$C = 120x + 45\left(\frac{10}{x}\right) = 120x + 450x^{-1}$$

$$C' = 120 - 450x^{-2} = 120 - \frac{450}{x^2}$$

C' undef. when $x=0$

$$x^2 \left(120 - \frac{450}{x^2} \right) = 0 \cdot x^2 \quad x^2 = \frac{450}{120}$$

$$120x^2 - 450 = 0 \quad x = \pm \sqrt{\frac{15}{4}}$$

$$x = \sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$$

$$y = \frac{10}{\frac{\sqrt{15}}{2}} = \frac{20}{\sqrt{15}}$$

each pen $\frac{\sqrt{15}}{2} \times \frac{20}{\sqrt{15}}$ ft
min. cost \$ 464.76

$$\begin{array}{c} \sqrt{\frac{15}{4}} \\ \times \\ \sqrt{\frac{15}{4}} \\ \hline 1.936 \end{array} \leftarrow \text{rel. min.}$$

Example 5 A rectangular box with open top is to be constructed from a rectangular piece of cardboard 12 inches by 18 inches by cutting out equal squares from each corner and folding up the resulting flaps. Find the dimensions of the box that will have the maximum volume.