Optimization notes Day 1

Example 1 Find two positive numbers whose product is 192, and the sum of the first plus three times the second is a minimum.

Let $x = 1^{st}$ #

$y = 2^{nd}$ #

minimize $S = x + 3y$

$xy = 192$

$y = \frac{192}{x}$

$S = x + 3\left(\frac{192}{x}\right) = x + \frac{576}{x} \rightarrow 576x^{-1}$

$S' = 1 - 576x^{-2}$

$1 - \frac{576}{x^2}$

undef when $x = 0$

$x^2\left(1 - \frac{576}{x^2}\right) = 0 \cdot x^2$

$x^2 - 576 = 0$

$x^2 = 576$

$x = \pm 24$

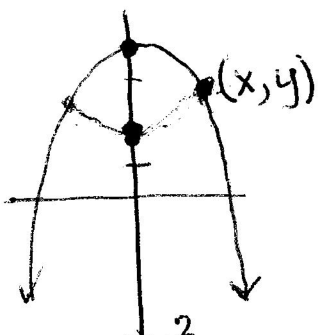
$\begin{array}{ccccc} & + & - & - & + \\ \hline & -24 & 0 & 24 & \end{array}$

rel min

$x = 24$

$y = \frac{192}{24} = 8$

Example 2 Which points on the graph of $y = 4 - x^2$ are closest to the point (0,2)?



minimize the distance between $(0,2)$ and (x,y)

$d = \sqrt{(x-0)^2 + (y-2)^2}$

$d = \sqrt{x^2 + (\underbrace{4 - x^2 - 2}_{2-x^2})^2}$

$d = \sqrt{x^2 + 4 - 4x^2 + x^4}$

$d = \sqrt{x^4 - 3x^2 + 4}$

$\frac{1}{2}(x^4 - 3x^2 + 4)^{-1/2}(4x^3 - 6x)$

deriv: $4x^3 - 6x = 0$

$2x(2x^2 - 3) = 0$

$2x = 0 \quad 2x^2 - 3 = 0$

$x = 0 \quad x = \pm\sqrt{\frac{3}{2}}$

$\begin{array}{ccccc} - & + & - & + \\ \hline -\sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & \end{array}$

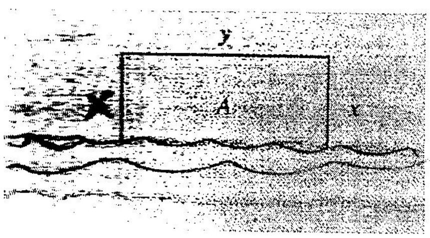
rel min

$y = 4 - \left(-\sqrt{\frac{3}{2}}\right)^2 = 2.5$

$y = 4 - \left(\sqrt{\frac{3}{2}}\right)^2 = 2.5$

$\left(-\sqrt{\frac{3}{2}}, 2.5\right)$ and $\left(\sqrt{\frac{3}{2}}, 2.5\right)$

Example 3 A farmer wants to fence in a rectangular region. A river runs along one of the sides of the region, so the farmer only needs to put the fence on three of the sides. If the farmer has 200 yards of fencing to use, find the largest possible area which can be enclosed by the fence. What are the dimensions of the region?



maximize $A = xy \qquad 2x + y = 200$

$y = 200 - 2x$

$A = x(200 - 2x) = 200x - 2x^2$

$A' = 200 - 4x = 0$

$-4x = -200$

$x = 50$

$\begin{array}{ccc} + & & - \\ \hline & 50 & \end{array}$

rel max

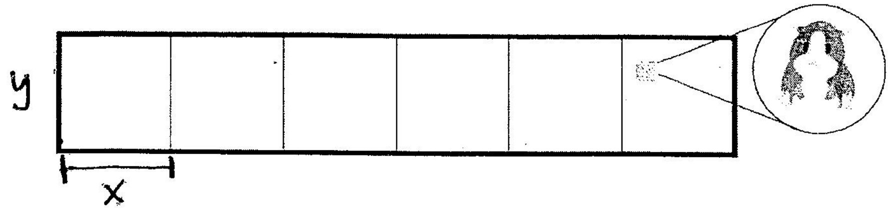
$y = 200 - 2(50) = 100$

$\boxed{50 \text{ yd by } 100 \text{ yd}}$

max area $= 50 \cdot 100$

$= 5000 \text{ yd}^2$

Example 4 Your dream of becoming a hamster breeder has finally come true. You are constructing a set of rectangular pens in which to breed your furry friends. The overall area you are working with is 60 square feet, and you want to divide the area up into six pens of equal size as shown below.



The cost of the outside fencing is \$10 a foot. The inside fencing costs \$5 a foot. You wish to minimize the cost of the fencing. Find the exact dimensions of each pen that will minimize the cost of the breeding ground. What is the total cost?

each pen $= 10\text{ ft}^2$ $\quad xy = 10 \quad y = \frac{10}{x}$

$C = 10(2y + 12x) + 5(5y) = 20y + 120x + 25y = 45y + 120x$

$C = 45\left(\frac{10}{x}\right) + 120x = 450x^{-1} + 120x$

$C' = -450x^{-2} + 120$

$x^2\left(\frac{-450}{x^2} + 120\right) = (0)x^2$ undef when $x = 0$

$-450 + 120x^2 = 0$

$120x^2 = 450$

$x^2 = \frac{15}{4}$

$x = \pm\sqrt{\frac{15}{4}} = \pm\frac{\sqrt{15}}{2}$

$y = \frac{10}{\frac{\sqrt{15}}{2}} = \frac{20}{\sqrt{15}}$

+ | − | +

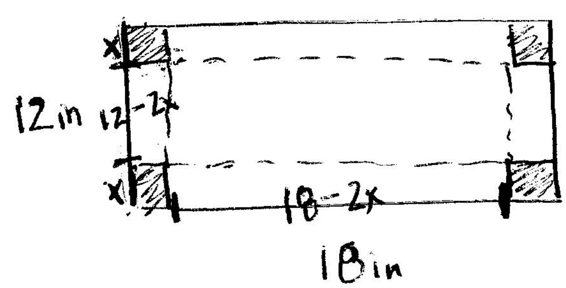
$\cancel{-\frac{\sqrt{15}}{2}}$ $\quad \cancel{0} \quad \frac{\sqrt{15}}{2}$

min

$\boxed{\frac{\sqrt{15}}{2}\text{ ft by } \frac{20}{\sqrt{15}}\text{ ft}}$

$\boxed{\text{cost} = \$464.76}$

Example 5 A rectangular box with open top is to be constructed from a rectangular piece of cardboard 12 inches by 18 inches by cutting out equal squares from each corner and folding up the resulting flaps. Find the dimensions of the box that will have the maximum volume.



$V = (18 - 2x)(12 - 2x)(x)$

$V = (216 - 36x - 24x + 4x^2)(x)$

$V = 216x - 60x^2 + 4x^3$

$V' = 216 - 120x + 12x^2$

$V' = 12(18 - 10x + x^2)$

$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(18)}}{2(1)}$

$x = \frac{10 \pm \sqrt{28}}{2} = \frac{10 \pm 2\sqrt{7}}{2} = 5 \pm \sqrt{7}$

+ | − | +

$5 - \sqrt{7}$ $\quad 5 + \sqrt{7}$

2.354 $\quad$ 7.646

max

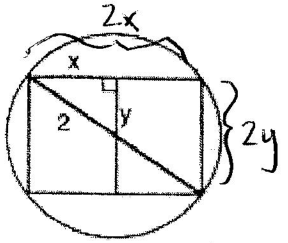
ht $= 5 - \sqrt{7}$ in

$l = 18 - 2x = 13.292$ in

$w = 12 - 2x = 7.292$ in

Optimization notes Day 2

Example 1 A rectangle is inscribed in a circle of radius 2 m. Find the maximum area of this rectangle.



eqn of circle: $x^2+y^2=2^2$

$x^2+y^2=4 \Rightarrow y^2=4-x^2$

$y=\pm\sqrt{4-x^2}$

$A_{rect}=(2x)(2y)=4xy$

$A=4x\left(\sqrt{4-x^2}\right)$

$A'=4x\cdot\frac{1}{2}(4-x^2)^{-1/2}(-2x)+\sqrt{4-x^2}\cdot 4$

$A'=\dfrac{-4x^2}{\sqrt{4-x^2}}+4\sqrt{4-x^2}=\dfrac{-4x^2+4(4-x^2)}{\sqrt{4-x^2}}=\dfrac{-8x^2+16}{\sqrt{4-x^2}}$

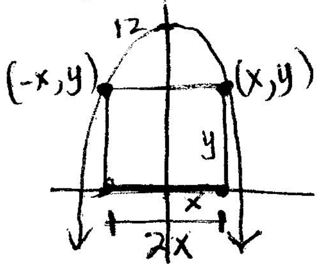
$-8x^2+16=0$
$x^2=2$
$x=\pm\sqrt{2}$

A' undef. when $x=2,-2$

$-2 \quad -\sqrt{2} \quad \sqrt{2} \quad 2$; $+ \quad -$

max when $x=\sqrt{2} \Rightarrow A=4\sqrt{2}\sqrt{4-(\sqrt{2})^2}=\boxed{8\ m^2}$

Example 2 A rectangle has a base on the x-axis and its two upper corners are on the parabola $y=12-x^2$. What is the largest possible area of the rectangle?



max. $A=2xy=2x(12-x^2)=24x-2x^3$

$A'=24-6x^2=0$

$-6x^2=-24$

$x^2=4$

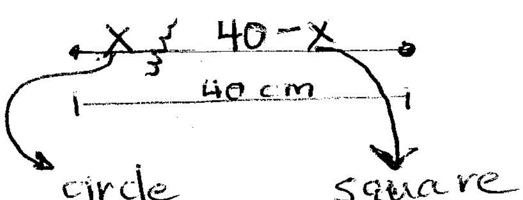
$x=\pm 2$

$\cancel{-2} \quad 2$; $+ \quad -$; max

$x=2 \quad y=12-2^2=8$

$A=2(2)(8)=\boxed{32}$

Example 3 A piece of wire 40 cm long is to be cut into 2 pieces to form a circle and a square. Find the lengths of each piece that cause the sum of the area of the circle and the area of the square to be a minimum.



$C=x=2\pi r \qquad P=40-x$

$r=\dfrac{x}{2\pi} \qquad \text{Side}=\dfrac{40-x}{4}=10-\frac{1}{4}x$

minimize Areas

$A=\pi r^2=\pi\left(\dfrac{x}{2\pi}\right)^2=\pi\cdot\dfrac{x^2}{4\pi^2}=\dfrac{x^2}{4\pi}$

$A=s^2=\left(10-\frac{1}{4}x\right)^2$

$S=\dfrac{x^2}{4\pi}+\left(10-\frac{1}{4}x\right)^2$

$S'=\dfrac{1}{4\pi}\cdot 2x+2\left(10-\frac{1}{4}x\right)^1\left(-\frac{1}{4}\right)$

$\dfrac{1}{2\pi}x-5+\dfrac{1}{8}x=0$

$x\left(\dfrac{1}{2\pi}+\dfrac{1}{8}\right)=5$

$x=\dfrac{5}{\frac{1}{2\pi}+\frac{1}{8}}=\dfrac{5}{\frac{4+\pi}{8\pi}}$

$x=\dfrac{40\pi}{4+\pi} \qquad 5\cdot\dfrac{8\pi}{4+\pi}$

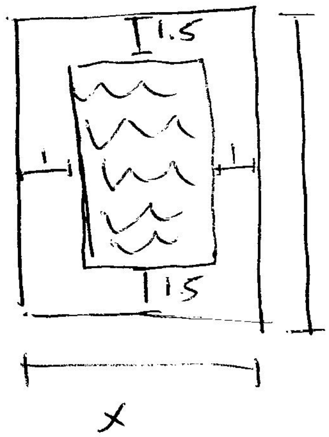
$- \quad +$; $\dfrac{40\pi}{4+\pi}$ 17.596

10

$x=\dfrac{40\pi}{4+\pi}$ cm

other: $40-\dfrac{40\pi}{4+\pi}$ cm

Example 4 A rectangular page is to have 24 square inches of print. The top and bottom margins are 1.5 inches each. The side margins are 1 inch each. Find the dimensions of the page so that the least amount of paper is used.



min. $A = xy$

$A_{print} = (x-2)(y-3) = 24$

$y - 3 = \frac{24}{x-2}$

$y = 3 + \frac{24}{x-2}$

$A = x\left(3 + \frac{24}{x-2}\right)$

$A = x\left(\frac{3(x-2)+24}{x-2}\right)$

$A = x\left(\frac{3x-6+24}{x-2}\right) = \frac{3x^2+18x}{x-2}$

$$A' = \frac{(x-2)(6x+18) - (3x^2+18x)(1)}{(x-2)^2}$$

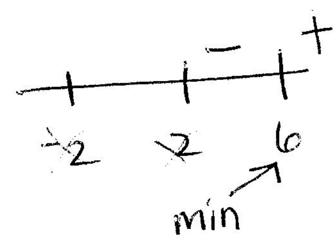
$$A' = \frac{3x^2-12x-36}{(x-2)^2}$$ undef at $x=2$

$3x^2 - 12x - 36 = 0$

$3(x^2 - 4x - 12) = 0$

$3(x-6)(x+2) = 0$

$x = 6, -2$



$x = 6$

$y = 3 + \frac{24}{6-2} = 9$

$\boxed{6\text{ in by }9\text{ in}}$