Pre-Calculus

Notes(7.4)-----Partial Fractions (with non-repeated linear factors)

Adding rational functions involves finding a common denominator, rewriting each, fraction so that it has that denominator, then adding.

For example: $\dfrac{-1}{x+2}+\dfrac{2}{x-6}=\dfrac{-1(x-6)}{(x+2)(x-6)}+\dfrac{2(x+2)}{(x+2)(x-6)}$

$$=\frac{-x+6+2x+4}{(x+2)(x-6)}=\frac{x+10}{x^2-4x-12}$$

In Calculus it is frequently easier to work with a sum of several simple fractions than it is to work with a single more complex fraction. For this reason it is important that we be able to "undo" the above process. This is commonly referred to as **DECOMPOSING** the fraction into **PARTIAL FRACTIONS**. Since the LCD we find is made up of factors of the denominators, we must find these factors to decompose. In order for the process to work correctly, we must factor the denominator as much as possible.

This will leave you with ONLY

1 – Linear factors
2 – Irreducible quadratic factors (quadratics with complex roots) "IQF"
3 – Powers of these

1st Step: Make sure the degree of the numerator is less than the degree of the denominator. If it isn't, use long division first.
2nd Step: Find the possible LCD for the decomposition (the factors)
3rd Step: Write the original fraction as a sum of fractions with these possible denominators & unknown numerators

Factor in denominator	Term in partial fraction decomposition
* $ax + b$---------Distinct Linear Factor	$\dfrac{A}{ax+b}$
$(ax + b)^k$---------Repeated Linear Factor	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\dots+\dfrac{A_k}{(ax+b)^k}$
$ax^2 + bx + c$---------Distinct Irreducible Quadratic Factor	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2 + bx + c)^k$---------Repeated Irreducible Quadratic Factor	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

4th Step: Clear the fractions (using the unknowns).
5th Step: Simplify.
6th Step: Equate the coefficients. (Create a system of equations)
7th Step: Solve the system and substitute these values back into the unknowns.

NOTE: All examples use distinct, linear factors.

Directions: Find the partial fraction decomposition of the following:

Ex 1) $\dfrac{x+10}{x^2-4x-12}=\dfrac{x+10}{(x-6)(x+2)}=\dfrac{A}{x-6}+\dfrac{B}{x+2}$

$x+10=A(x+2)+B(x-6)$

$x=-2$: $8=A(0)+B(-8)$
$B=-1$

$x=6$: $16=A(8)+B(0)$
$A=2$

$$\boxed{\frac{2}{x-6}+\frac{-1}{x+2}}$$

Ex 2) $\dfrac{2x}{x^2-1}=\dfrac{2x}{(x+1)(x-1)}=\dfrac{A}{x+1}+\dfrac{B}{x-1}$

$2x=A(x-1)+B(x+1)$

$x=1$: $2=A(0)+B(2)$
$B=1$

$x=-1$: $-2=A(-2)+B(0)$
$A=1$

$$\boxed{\frac{1}{x+1}+\frac{1}{x-1}}$$

Ex 3) $\dfrac{x+3}{2x^2-9x-5} = \dfrac{x+3}{(2x+1)(x-5)} = \dfrac{A}{2x+1} + \dfrac{B}{x-5}$

$x+3 = A(x-5) + B(2x+1)$

$x=5$: $8 = A(0) + B(11)$

$B = \frac{8}{11}$

$x=-\frac{1}{2}$: $\frac{5}{2} = A\left(-\frac{11}{2}\right) + B(0)$

$A = \dfrac{\frac{5}{2}}{-\frac{11}{2}} = -\frac{5}{11}$

$$\boxed{\dfrac{-\frac{5}{11}}{2x+1} + \dfrac{\frac{8}{11}}{x-5}}$$

Ex 4) $\dfrac{2x+1}{x^3+x^2-6x} = \dfrac{A}{x} + \dfrac{B}{x+3} + \dfrac{C}{x-2}$

$x(x^2+x-6)$

$x(x+3)(x-2)$

$$\boxed{\dfrac{-\frac{1}{6}}{x} + \dfrac{-\frac{1}{3}}{x+3} + \dfrac{\frac{1}{2}}{x-2}}$$

$2x+1 = A(x+3)(x-2) + B(x)(x-2) + C(x)(x+3)$

$x=0$: $1 = A(3)(-2) + B(0)(-2) + C(0)(3)$

$1 = A(-6)$

$A = -\frac{1}{6}$

$x=2$: $5 = A(5)(0) + B(2)(0) + C(2)(5)$

$5 = 10C$

$C = \frac{1}{2}$

$x=-3$: $-5 = A(0)(-5) + B(-3)(-5) + C(-3)(0)$

$-5 = 15B$ $B = -\frac{1}{3}$

Ex 5) $\dfrac{15x^2-11x-5}{x(x+1)(2x-5)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{2x-5}$

$15x^2-11x-5 = A(x+1)(2x-5) + B(x)(2x-5) + C(x)(x+1)$

$x=-1$: $21 = A(0)(-7) + B(-1)(-7) + C(-1)(0)$

$21 = 7B$

$B = 3$

$x=0$: $-5 = A(1)(-5) + B(0)(-5) + C(0)(1)$

$-5 = -5A$

$A = 1$

$x=\frac{5}{2}$: $61.25 = A(3.5)(0) + B(\frac{5}{2})(0) + C(\frac{5}{2})(\frac{7}{2})$

$61.25 = \frac{35}{4}C$

$C = 7$

$$\boxed{\dfrac{1}{x} + \dfrac{3}{x+1} + \dfrac{7}{2x-5}}$$

Ex 6) $\dfrac{x^3-4x-10}{x^2-x-6}$ hint: divide first!!!

$$\begin{array}{r} x+1 \\ x^2-x-6 \overline{\smash{)}\, x^3+0x^2-4x-10} \\ -(x^3-x^2-6x) \\ \hline x^2+2x-10 \\ -(x^2-x-6) \\ \hline 3x-4 \end{array}$$

$x+1+\dfrac{3x-4}{x^2-x-6}$

$(x-3)(x+2)$

$\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$

$3x-4 = A(x+2) + B(x-3)$

$x=-2$: $-10 = A(0) + B(-5)$

$B = 2$

$x=3$: $5 = A(5) + B(0)$

$A = 1$

$$\boxed{x+1+\dfrac{1}{x-3}+\dfrac{2}{x+2}}$$